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Calculus and Mechanics on Two-Point Homogenous Riemannian Spaces

Springer

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Preface

Mathematics develops both due to demands of other sciences and due to its internal logic. The latter fact explains the attention of mathematicians to many problems, which are in close connection with basic mathematical notions, even if these problems have no direct practical applications.

It is well known that the space of constant curvature is one of the basic geometry notion [208], which induced the wide field for investigations. As a result there were found numerous connections of constant curvature spaces with other branches of mathematics, for example, with integrable partial differential equations [36, 153, 189][1] and with integrable Hamiltonian systems [141]. Geodesic flows on compact surfaces of a constant negative curvature (with the genus $\geqslant 2$) generate many test examples for ergodic theory (see also [183] and the bibliography therein). The hyperbolic space $\mathbf{H}^3(\mathbb{R})$ is the space of velocities in special relativity (see Sect. 7.4.1) and also arises as space-like sections in some models of general relativity.

Long before the creation of general relativity the founders of the hyperbolic geometry, Lobachevsky and Bolyai, made initial attempts to transfer the Newtonian mechanics onto the hyperbolic space. They proposed the analog of the Newtonian potential for this space. In 1885, Killing in his important paper [87] gave a detailed consideration of the one-particle motion in the analog of the Newtonian potential on the three-dimensional sphere $\mathbf{S}^3$ and found analogs of three Kepler laws for this problem. In Liebmann papers [103] and [105], the Killing results were transferred onto the hyperbolic space $\mathbf{H}^3(\mathbb{R})$.

Soon after the appearance of quantum mechanics Schrödinger [156], Infeld, and Schild [73] considered the corresponding quantum-mechanical one-body problems for the Newtonian (Coulomb) potentials in spaces $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ and found corresponding energy levels.

However, these results did not become widely known and at the end of the 20th century were rediscovered many times by some authors. Even the recent book on classical one-body problem in spaces $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ [204] ignores the results of Killing and Liebmann, erroneously ascribing them to papers [37] and [93] published at the end of 20th century. Section 6.4 of present book gives

[1] The author also made his modest contribution in this subject [167].

the detailed description of the history of the one-body problem in spaces of constant curvature and thus fills the existing gap in the modern literature.

The two-body problem with central interaction in constant curvature spaces was considered firstly by the author: the classical one in [160] and the quantum mechanical one in [162]. In Euclidean space this problem is reduced to the one-body problem in a central potential after separating the center of mass motion. Due to the absence of Galilei transformations the situation for the constant curvature spaces is different [160]. The two-body problem is invariant with respect to the isometry group, but for non-Euclidean space this group is not wide enough to imply the integrability of this problem in any sense.

The natural problem of finding central potentials corresponding to integrable two-body problems is far from its solution now. At the first glance it seems a bit strange due to the diversity of methods in the theory of integrable dynamical systems, but a more closer look at these methods shows that they are not adapted for systems under consideration.

Indeed, some methods are aimed at an artificial construction of new integrable systems; other methods explain from new points of view the integrability of old integrable systems. For studying the integrability of a concrete dynamical system one can use only "old" methods: the numerical construction of Poincaré surface of section, the Painlevé test, and lucky guesses, concerning the form of additional integrals if they exist. However, in our situation the latter methods encounter great difficulties.

The numerical construction of Poincaré surfaces of section for the classical two-body system is possible only for concrete potentials of interaction. Therefore, one could not find a potential, corresponding to integrable case without a lucky guess (see also the discussion in Sect. 7.3.2). The Painlevé test (see for example [191]) practically is suitable only for differential equations having a polynomial form. It is not valid for the two-body problem in spaces of constant curvature. Lucky guesses, if appear, do it usually at once and only in relatively simple situations. In another case one should wait for the elaboration of a refined technique aimed at a concrete subject.[2]

Clearly, all reasons above, concerning the integrability of the two-body problem on constant curvature spaces, are purely heuristic and do not exhaust this problem. Note in this connection the negative results concerning the nonintegrability of the restricted [112, 213, 214] and nonrestricted [171] two-body problem in spaces of constant curvature for Newton and oscillator potentials.

The natural spaces for the two-body problem are two-point homogeneous Riemannian spaces, because a distance between bodies in these spaces is the only invariant of their location with respect to the isometry group. Simply connected spaces of a constant curvature and real projective spaces are particular cases of two-point homogeneous Riemannian spaces.[3] The two-body system in these spaces has the radial degree of freedom and degrees being described through the isometry group.

[2] Recall for example the history of the Fermat last theorem [147].

[3] Below by a two-point homogeneous space we shall mean any space of this kind except Euclidean one.

However, the problem of finding an explicitly invariant form of the two-body Hamiltonian on two-point homogeneous spaces turned out to be a difficult problem, which was solved by stages in author's papers [166, 160, 162, 163]. At last, a general formula of the two-body Hamiltonian valid for all two-point homogeneous spaces was found in [169] on the base of a special expansion of a Lie algebra $\mathfrak{g}$, corresponding to the isometry group G, into a direct sum of subspaces.

This approach uses the Helgason theory of invariant differential operators [66, 67]. In the quantum mechanical case it leads to the representation of the two-body Hamiltonian H through a radial differential operator and generators of the algebra $\mathrm{Diff}_I(Q_S)$ of invariant differential operators on the unit sphere bundle Q_S over a two-point homogeneous space Q. These generators are polynomial with respect to a base of the algebra $\mathfrak{g}$. This representation of H enables one to find separate spectral differential equations for the two-body problem on compact two-point homogeneous spaces. For Coulomb and oscillator potentials on spheres $\mathbf{S}^n$ some of these equations can be reduced to the hypergeometric equation and thus be solved in an explicit form. Therefore, Coulomb and oscillator quantum mechanical two-body problems on $\mathbf{S}^n$ are *quasi-exactly solvable* [162, 184].

Using the correspondence between classical Hamiltonian functions and quantum mechanical Hamiltonians, one can derive the explicitly invariant form of the two-body Hamiltonian function. Generators of the algebra $\mathrm{Diff}_I(Q_S)$ are replaced by generators of the corresponding graded algebra $\mathrm{gr}\,\mathrm{Diff}_I(Q_S)$, which is isomorphic to the Poisson algebra of invariant functions on the cotangent bundle T^*Q_S. Using this form of the Hamiltonian function one can compare different approaches [47, 48, 128, 152, 215] to the definition of the center of mass for two particles on constant curvature spaces. This form is also convenient for the Hamiltonian reduction of the two-body problem and for proving the absence of particles' collision on infinite time interval under some additional conditions.

Note that these investigations require various geometrical, algebraic, and analytical methods. For analyzing general situations we use here the coordinate free language, preferably in terms of corresponding Lie algebras. Indeed, for curved spaces, especially of a high dimension (for example dimension n), the existing symmetry is often hidden in cumbersome coordinate expressions. The manipulation with such expressions becomes very laborious and frequently impossible even with the help of computer algebra systems. On the other hand, expressions of invariant differential operators through base elements of a Lie algebra are polynomial with constant coefficients in noncommutative variables. Manipulations with such polynomials are much easier than coordinate evaluations.

Some results of the present book are of more general interest and can be used in other researches. These are the expression of the Laplace–Beltrami operator through a moving frame, particularly through Killing vector fields, the description of the reduced cotangent bundle over a G-homogeneous space in terms of orbits of the Ad^*_G-action, and the description of the algebra $\mathrm{Diff}_I(Q_S)$ for a two-point homogeneous space Q through generators and relations.

Chapters 1–4 describe the geometry results necessary for studying the two-body problem on two-point homogeneous spaces. The classification of these spaces are in Chap. 1. There are also models of compact two-point homogeneous spaces as submanifolds of Euclidean spaces or its factor spaces, different models of real hyperbolic spaces $\mathbf{H}^n(\mathbb{R})$, $n \geqslant 2$, and the description of the transition from compact to noncompact two-point homogeneous spaces in terms of corresponding Lie algebras.

Necessary data on differential operators are in the second chapter. Section 2.1 contains basic notions of the theory of invariant differential operators on smooth manifolds. The expression of the Laplace–Beltrami operator is derived in Sect. 2.2. Basic facts on self-adjointness of abstract differential operators and conditions sufficient for the self-adjointness of Schrödinger operators on Riemannian manifolds are described in Sect. 2.3. The last Sect. 2.4 of Chap. 2 contains the general scheme of the quantum-mechanical reduction.

The third chapter deals with algebras $\mathrm{Diff}_I(Q_S)$ of invariant differential operators on unit sphere bundles Q_S over two-point homogeneous spaces Q. There are found the description of these algebras in terms of generators and relations. All such systems of generators contain one generator D_0 of the first order. Its kernel is studied in Sect. 3.6. Also, there are found some elements from centers of these algebras.

Chapter 4 contains basic facts concerning Hamiltonian dynamical systems with symmetry and the correspondence between classical and quantum-mechanical systems. In particular, the noncommutative integrability and the momentum map are discussed here. The special symplectomorphism between a reduced cotangent bundle of a homogeneous manifold and some factor space of a coadjoint orbit of a corresponding Lie group is constructed in Sect. 4.3.4.

Chapter 5 deals with the problem of finding an explicitly invariant expression for the two-body Hamiltonian on a two-point homogeneous space Q through a radial differential operator and generators of the algebra $\mathrm{Diff}_I(Q_S)$.

The one-body problem in a central potential is discussed in Chap. 6. In Sect. 6.1 the noncommutative integrability of the classical one-body problem on a general two-point homogeneous space is proved, which seems to be a new result. For spaces of constant curvature there are given more detailed results both in classical and quantum cases. These results are known, but are collected together for the first time.

The classical case includes the discussion of the genesis of Bertrand potentials, the description of particle trajectories in these potentials, and their relations with conics in constant curvature spaces. In the quantum case there are given explicit formulas for one-particle energy levels and eigenfunctions, corresponding to Bertrand potentials in these spaces.

Also, in Sect. 6.4 there is a historical survey of one- and two-body motions in central potentials in spaces of a constant curvature, containing relevant references to many papers from the early beginning of the non-euclidean geometry till the present time.

The expression for the two-body quantum Hamiltonian from Chap. 5 is transformed into the two-body Hamiltonian function of the corresponding classical system in Chap. 7. The problem of searching of nontrivial integrals of motion for different potentials is discussed in Sect. 7.3. Also, the absence of particles' collision for some potentials and initial conditions is proved. The found expression for the two-body Hamiltonian function is considered in Sect. 7.4 with respect to the center-of-mass problem in two-point homogeneous spaces. Different existing definitions of the center-of-mass for constant curvature spaces are discussed. It is shown that all of them have flaws in comparison with the center of mass concept in Euclidean space. Reduced classical two-body systems in spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$ are classified in Sect. 7.5.

Chapter 8 deals with the quantum two-body problem in compact two-point homogeneous spaces. It is shown that some infinite energy level series can be found from separate ordinary differential equations of the second order. All such equations are found for spheres $\mathbf{S}^n$; then they are reduced to the hypergeometric equation for Coulomb and oscillator potentials and corresponding energy levels series are found in explicit form. Thus, the quasi-exactly solvability of the two-body problem for Coulomb and oscillator potentials on spheres is shown. Difficulties of using this approach for noncompact two-point homogeneous spaces are discussed.

There are also four appendices in the book. The first one demonstrates the technique of calculating commutative relations for generators of differential operator algebras from Chap. 3. The second appendix contains basic facts on Fuchsian differential equations, especially on Riemann, hypergeometric, and Heun ones. In the third appendix there are some facts concerning orthogonal complex Lie algebras and their representations. Some unsolved problems arising from the book content are listed in the last appendix.

Prerequisites from differential geometry can be found in [17, 32, 34, 56, 63, 64, 92, 143, 208]; from modern theory of Hamiltonian systems in [8, 32, 58, 114, 116, 181, 193]; from the theory of Lie groups and their actions on smooth manifolds in [2, 3, 13, 31, 65, 66, 88, 134, 142, 158, 199]; from representation theory [53, 60, 135, 212]; from functional analysis in [44, 85, 144]; and many other sources.

If one is interested in a brief introduction into the one-body problem on constant curvature spaces he or she can read Sects. 1.3.3 and 6.2 for the classical case and Sects. 1.3.3, 2.3 and 6.3 for the quantum one.

The author tried to make the bibliography as complete as possible only in respect of papers, concerning one- and two-body mechanics on two-point homogeneous spaces, particularly on spaces of constant curvature, except of geodesic flows. A survey on the latter subject can be found in [25].

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Moscow *Alexey V. Shchepetilov*
February 2006

Contents

Glossary

Sets

$\mathbb{N}$	is the set of natural numbers
$\mathbb{R}_+$	is the set of positive real numbers
$(\alpha, \dots, \omega)$	denotes a set of objects $\alpha, \dots, \omega$
pt	denotes a one point set

Spaces

$\mathcal{L}^2(M, d\nu)$	is the Hilbert space of complex-valued functions on M, square integrable w.r.t. a measure ν
$\mathcal{L}^2_{\text{loc}}(M, d\nu)$	is the set of complex-valued functions on M, locally square integrable w.r.t. a measure ν
$W^{k,l}_{\text{loc}}(M^n, d\mu)$	37[4]
Q	denotes a two-point homogeneous Riemannian space, different from Euclidean one
$M_{\mathbf{S}}$	denotes the unit sphere bundle over a Riemannian space M
M/G	denotes a factor space of a space M with respect to an action of a group G on it
$\text{span}(e_1, \dots, e_n)$	denotes the linear span of elements $e_1, \dots, e_n$ from some linear space
	for a linear space L the space L^* is dual to L
	for a subspace L' of a linear space L the subspace $\text{ann}\, L' \subset L^*$ is the annihilator of L', i.e., $\text{ann}\, L' := (\alpha \in L^* \mid \alpha(L') = 0)$

Algebras and Groups

$\mathbb{R}$	is the field of real numbers
$\mathbb{C}$	is the field of complex numbers
$\mathbb{H}$	is the algebra of quaternions

[4] A number after notation refers to a page, where this notation is described.

$\mathbb{C}a$	is the algebra of octonions (the Cayley algebra)
$C^\infty(M)$	is the algebra of smooth real-valued functions on a smooth manifold M
$C^\infty_c(M)$	is the subalgebra of $C^\infty(M)$, consisting of functions with a compact support
$C^\infty(M,\mathbb{C})$	is the algebra of smooth complex-valued functions on M
$C^\infty_c(M,\mathbb{C})$	is the subalgebra of $C^\infty(M,\mathbb{C})$, consisting of functions with a compact support
$\mathcal{P}(T^*M)$	is the algebra of smooth real-valued functions on a cotangent bundle T^*M, polynomial on fibers
$\mathcal{X}(M)$	is the infinite-dimensional Lie algebra of smooth vector fields on M; also $\mathcal{X}(M)$ is a module over $C^\infty(M)$
$U(\mathfrak{g})$	is the universal enveloping algebra for a Lie algebra $\mathfrak{g}$
$\mathrm{LDiff}(G)$, $\mathrm{RDiff}(G)$, $\mathrm{LRDiff}(G)$	are respectively algebras of left-, right- and biinvariant differential operators on a Lie group G
$\mathrm{Diff}_G(M)$	is the algebra of G-invariant differential operators on a G-homogeneous space M
$\mathrm{LDiff}_K(G)$	is the algebra of G left-invariant and K right-invariant differential operators on G, where K is a subgroup of G
$\mathrm{O}(1,n)$, $\mathrm{O}_0(1,n)$	13
$W_{\mathcal{F}}(x)$	75
$S_p(\mathfrak{g})$	76

Operations

$\circ$	denotes the Jordan multiplication in the algebra $\mathfrak{h}_3(\mathbb{C}a)$; in other cases it denotes the composition of two operations
$\pounds_\xi T$	is the Lie derivative of a tensor field T along a vector field ξ
∇	is the Levi-Civita connection on a Riemannian manifold
$\operatorname{grad} f$	is the gradient of a function f on a Riemannian manifold
ad_X	denotes the adjoint action of an element X from a Lie algebra
Ad_q	denotes the adjoint action of an element q from a Lie group
Ad^*_q	denotes the coadjoint action of an element q from a Lie group 77
$[A,B]$	denotes the commutator in algebras, in particular the commutator of vector fields as operators, acting on functions
$\{A,B\}$	denotes the anticommutator in algebras
$[\varphi,\psi]_P$	denotes the Poisson brackets of functions φ and ψ on a Poisson manifold
$\langle\cdot,\cdot\rangle$	denotes a scalar (inner) product
$\operatorname{im}\lambda$	is the image of a map λ
λ^{-1}	is the inverse map (generally multivalued) for a map λ
id	is the identity map

π_i	denotes different projections π_1 is a projection of a group onto its homogeneous space 8, 10, 24, 97, π_2 – 33, π_3 – 54, π_4 – 98, $\tilde{\pi}_1, \tilde{\pi}_2$ – 113,
$d\pi$ and π_*	denote the differential of a map π
$d\pi^*$	denotes the codifferential of a map π
$\oplus$	denotes a direct sum of linear spaces, until indicated otherwise
$\mathrm{Kil}_{\mathfrak{g}}$	is the Killing form for a Lie algebra $\mathfrak{g}$

Miscellania

$\mathbf{i}, \mathbf{j}, \mathbf{k}$	are quaternion complex units
$\dim_{\mathbb{K}}$	denotes a dimension of some object over a field $\mathbb{K}$
$(z_1 : \ldots : z_{n+1})$	are homogeneous coordinates of a point from a projective space
$\operatorname{Re} z, \operatorname{Im} z$	are respectively real and imaginary parts of an element $z \in \mathbb{C}, \mathbb{H}, \mathbb{C}a$
$A \setminus B$	denotes the set subtracting
$\operatorname{symb} D$	is the symbol of a differential operator D 104
$\mathfrak{S}_l$	denotes the group of all permutations of l elements

Until indicated otherwise, all manifolds, linear spaces, algebras, etc. are supposed to be real; smooth manifolds are supposed to be Hausdorff, paracompact and second countable.

Lie groups are denoted by capital Latin letters and their Lie algebras by corresponding small gothic letters. Also, small gothic letters denote linear subspaces of Lie algebras. Four series of simple classical complex Lie algebras are denoted as $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n, \mathfrak{D}_n$.

For a linear space V the symbol $T(V)$ denotes the tensor algebra without unit. We suppose also that the symmetric algebra $S(V)$ and the universal enveloping algebra $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ do not contain the unit element.

If a Lie group G acts in a linear space V, then its *invariant* means an invariant polynomial with arguments from V, i.e., an invariant element from the symmetric algebra $S(V^*)$ for the adjoint space V^*.

A scalar (inner) product in complex and quaternion linear spaces is supposed to be linear w.r.t. the second argument and conjugate linear w.r.t. the first one. A quaternion space is the right one w.r.t. quaternion multiplication.

A square root for the positive number is positive; for other numbers it is an arbitrary root.

Throughout the book by a polynomial with noncommutative arguments we mean an ordered one, i.e., each its monomial is an ordered product.

The standard abbreviations "iff" and "w.r.t." mean respectively "if and only if" and "with respect to".

1

Two-Point Homogeneous Riemannian Spaces

Euclidean space has many nice properties. It is a connected and simply connected topological space, i.e., its topology is simple. It is a Riemann space and from the point of view of Riemannian geometry it is flat, complete, homogeneous and isotropic. Moreover, a space with such properties is unique in every dimension. Therefore, Euclidean space is a suitable and historically the first area for constructing mechanical models.

If one does not insist on flatness of the physical spaces, but wants to conserve other nice properties described above (may be except of the property to be a simply connected space), then the next candidates for the role of a physical space are so called *two-point homogeneous spaces*, i.e., spaces in which any pair of points can be transformed by means of an appropriate isometry to any other pair of points with the same distance between them. The most well-known representatives of this class of Riemann spaces are simply connected *spaces of a constant sectional curvature*. Their isometry group reaches the maximal dimension (while fixing the dimension of a space) and they appear as space-like sections in some general relativity models.

In this chapter we start from the classification of two-point homogeneous Riemannian spaces, then describe a special expansion of algebras of their local isometries. Using this expansion and the homogeneity property the explicitly invariant form of the two-point Hamiltonian in these spaces will be obtain in the general way in Chap. 5. However, the studying more subtle properties of these spaces (e.g., algebras of invariant differential operators on the unit sphere bundles over these spaces) requires their models, which are described at the end of this chapter.

1.1 Classification

In the following, Q denotes a two-point homogeneous connected Riemannian space. The classification of these spaces can been found in [192, 206], (see also [118, 208]) and is as follows (everywhere $n \in \mathbb{N}$):

1. Euclidean space $\mathbf{E}^n$, $n \geqslant 1$;
2. the sphere $\mathbf{S}^n$, $n \geqslant 1$;

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3. the real projective space $\mathbf{P}^n(\mathbb{R})$, $n \geqslant 2$;
4. the complex projective space $\mathbf{P}^n(\mathbb{C})$, $n \geqslant 2$;
5. the quaternion projective space $\mathbf{P}^n(\mathbb{H})$, $n \geqslant 2$;
6. the Cayley projective plane $\mathbf{P}^2(\mathbb{C}a)$;
7. the real hyperbolic space (Lobachevski space) $\mathbf{H}^n(\mathbb{R})$, $n \geqslant 2$;
8. the complex hyperbolic space $\mathbf{H}^n(\mathbb{C})$, $n \geqslant 2$;
9. the quaternion hyperbolic space $\mathbf{H}^n(\mathbb{H})$, $n \geqslant 2$;
10. the Cayley hyperbolic plane $\mathbf{H}^2(\mathbb{C}a)$.

Note that there are isomorphisms in low dimensions: $\mathbf{P}^1(\mathbb{R}) \cong \mathbf{S}^1$, $\mathbf{P}^1(\mathbb{C}) \cong \mathbf{S}^2$, $\mathbf{P}^1(\mathbb{H}) \cong \mathbf{S}^4$, $\mathbf{P}^1(\mathbb{C}a) \cong \mathbf{S}^8$, $\mathbf{H}^1(\mathbb{C}) \cong \mathbf{H}^2(\mathbb{R})$, $\mathbf{H}^1(\mathbb{H}) \cong \mathbf{H}^4(\mathbb{R})$, $\mathbf{H}^1(\mathbb{C}a) \cong \mathbf{H}^8(\mathbb{R})$ [56, 66].

There are different equivalent approaches to the classification of these spaces. Recall that the *rank of a symmetric space* is the dimension of its maximal flat completely geodesic submanifold.

Theorem 1.1. *Let Q be a connected Riemannian space, G be the identity component of the isometry group for Q and K_x be a stationary subgroup of G for a point $x \in Q$. Then the following conditions 1, 2 are equivalent*

1. *Q is two-point homogeneous;*
2. *the action of the stationary subgroup K_x on all unit spheres in the tangent spaces T_xQ, $\forall x \in Q$ is transitive; in other words, Q is isotropic.*

These conditions together mean that the group G acts transitively on the set of all geodesics. Therefore, if any of these condition is satisfied, then all geodesics on the compact space Q are closed and have the same length.

Also, if a complete simply connected Riemannian space is a symmetric space of the rank one, then it is two-point homogeneous.

This result has been proved in [208] (lemma 8.12.1), [192, 206], see also references in [65] (p. 535). For compact two-point homogeneous Riemannian spaces (i.e., of types 2-6) the group G is compact and for other two-point homogeneous Riemannian spaces it is noncompact.

There are also two following results, characterizing some two-point homogeneous Riemannian spaces by its sectional curvatures $\varkappa$.

Theorem 1.2 (M. Berger, [34]). *Let M be a complete simply connected even-dimensional Riemannian manifold, all of whose sectional curvatures $\varkappa$ obey the inequality*

$$\frac{1}{4} \leqslant \varkappa \leqslant 1 \tag{1.1}$$

and whose diameter is π. Then M is isometric to a Riemannian symmetric space of rank one, which is also two-point homogeneous due to Theorem 1.1.

Theorem 1.3 ([34]). *Let M be a complete simply connected Riemannian manifold, all of whose sectional curvatures $\varkappa$ obey the inequality* (1.1). *Then M is homeomorphic to a sphere or is isometric to a Riemannian symmetric space of rank one, again two-point homogeneous.*

Useful models for all compact two-point homogeneous Riemannian spaces and the space $\mathbf{H}^n(\mathbb{R})$ are given below. There are known also models for spaces $\mathbf{H}^n(\mathbb{C})$ and $\mathbf{H}^n(\mathbb{H})$. The author is not aware if a model for the space $\mathbf{H}^2(\mathbb{C}a)$ is known.

Recall that a *rank* $\operatorname{rk}\mathfrak{g}'$ *of a semisimple Lie algebra* $\mathfrak{g}'$ (complex or real) is the dimension of its maximal commutative subalgebra. The rank $\operatorname{rk}\mathfrak{g}'$ coincides with the minimal codimension of Ad_G-orbits. Orbits of such codimension form an open dense subset in $\mathfrak{g}'$.

1.2 Special Expansion of the Lie Algebra of Infinitesimal Isometries for Two-Point Homogeneous Riemannian Spaces

Let now Q be a compact two-point homogeneous Riemannian space (i.e., a space of types 2-6), G be the identity component of the isometry group for Q and $\mathfrak{g}$ be the Lie algebra of G. We assume that some point $x_0 \in Q$ is fixed (the index x_0 will sometimes be omitted in the following), K is the stationary subgroup of G, corresponding to this point, and $\mathfrak{k} \subset \mathfrak{g}$ is the Lie algebra of K. All geodesics on Q are closed[1] and due to Theorem 1.1 have the same length equals $2\operatorname{diam} Q$, where $\operatorname{diam} Q$ is the maximal distance between two points of the space Q (i.e., $\operatorname{diam} Q$ is the diameter of Q). Put $R = 2\operatorname{diam} Q/\pi$ for the space $\mathbf{P}^n(\mathbb{R})$ and $R = \operatorname{diam} Q/\pi$ for the other compact two-point homogeneous Riemannian spaces. Then the maximal sectional curvature of all these spaces is R^{-2} and the minimal sectional curvature of the spaces $\mathbf{P}^n(\mathbb{C})$, $\mathbf{P}^n(\mathbb{H})$, $\mathbf{P}^2(\mathbb{C}a)$ is $(2R)^{-2}$ [65].

Denote by $\rho(x,y)$ the distance between points $x, y \in Q$ and let $\rho_\gamma(x,y)$ be the length of a smooth curve γ joining the points x and y. If γ is closed we denote by $\rho_\gamma(x,y)$ the distance of the shortest segment of γ joining x and y. For $x \in Q$ the subset $A_x \subset Q$, consisting of all points y for those $\rho(x,y) = \operatorname{diam} Q$, is called the *antipodal manifold* for x. For the sphere $\mathbf{S}^n$ the manifold A_x consists of one point, for the space $\mathbf{P}^n(\mathbb{K})$ it is isometric to $\mathbf{P}^{n-1}(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. For the space $\mathbf{P}^2(\mathbb{C}a)$ the manifold A_x is isometric to $\mathbf{S}^8$ [65, 67].

Proposition 1.1. *Let γ be a closed geodesic containing the point x_0 and a point $x_1 \in \gamma$ such that $\rho_\gamma(x_1, x_0) = \operatorname{diam} Q$. Then*

1. *$\rho_\gamma(x_0, x) = \rho(x_0, x)$ for every $x \in \gamma$, in particular $x_1 \in A_{x_0}$;*
2. *if a geodesic $\gamma_1 \neq \gamma$ contains x_0 and another point $x \in \gamma$, then $x = x_1$;*
3. *if $x \in \gamma$ and $0 < \rho(x_0, x) < \operatorname{diam} Q$, then the subgroup K_0 of the group G, consisting of all isometries conserving x_0 and x, conserves also all points of γ; thus K_0 does not depend on a choice of x if $0 < \rho(x_0, x) < \operatorname{diam} Q$.*

Proof. Throughout the proof we use the fact that in a complete connected Riemannian manifold a distance between any two points is realized by some

[1] Besides compact two-point homogeneous spaces there are other Riemannian spaces for which all geodesics are closed [16, 91].

geodesic and conversely any piecewise smooth curve, realizing this distance, is a geodesic [92, 208].

Let $x_2 \in A_{x_0}$. Then there is a geodesic γ' joining x_0 with x_2 such that $\rho_{\gamma'}(x_0, x_2) = \rho(x_0, x_2) = \operatorname{diam} Q$. Due to Theorem 1.1 there is an isometry $q \in K$, transforming γ' into γ. Therefore, it holds

$$\operatorname{diam} Q = \rho(x_0, x_2) = \rho_{\gamma'}(x_0, x_2) = \rho(x_0, qx_2) = \rho_{\gamma}(x_0, qx_2)\ .$$

This means $qx_2 = x_1$ and thus $\rho(x_0, x_1) = \rho_{\gamma}(x_0, x_1) = \operatorname{diam} Q$.

Suppose $x \in \gamma$ and $\rho_{\gamma}(x_0, x) < \operatorname{diam} Q$, but $\rho(x_0, x) < \rho_{\gamma}(x_0, x)$. Let the distance between x_0 and x is realized by a geodesic $\widetilde{\gamma}$. Then the piecewise geodesic, consisting of $\widetilde{\gamma}$ and the interval of γ between the points x and x_1, has the length less than $\operatorname{diam} Q$ that is impossible. This proves the first claim.

Prove the second claim. The assumption $\rho_{\gamma_1}(x_0, x) < \rho_{\gamma}(x_0, x)$ contradicts the first claim, already proved. Due to the equivalence of all geodesics in Q the inverse inequality $\rho_{\gamma_1}(x_0, x) > \rho_{\gamma}(x_0, x)$ is also impossible. Therefore $\rho_{\gamma_1}(x_0, x) = \rho_{\gamma}(x_0, x)$. If $x \neq x_1$, then there is the piecewise geodesic, consisting of two geodesic segment and joining the points x_0 and x_1, with the length $\operatorname{diam} Q$. This implies that $\rho(x_0, x_1) < \operatorname{diam} Q$, which contradicts the first claim. Thus, $x = x_1$.

Prove the last claim. If $q \in K_0$, then q transforms a segment $\widehat{\gamma}$ of γ, joining the points x_0 and x, into a geodesic segment, joining the same points. But due to the second claim there are no such geodesic segments except of $\widehat{\gamma}$ and $\gamma \backslash \widehat{\gamma}$. Since these segments have different lengths, it holds $q(\widehat{\gamma}) = \widehat{\gamma}$. Due to the same reasons it holds $q(\gamma \backslash \widehat{\gamma}) = \gamma \backslash \widehat{\gamma}$ that yields also $q(\gamma) = \gamma$.

Let γ_{K_0} be a subset of γ consisting of all K_0-fixed point. The continuity of the K_0-action on the space Q implies that γ_{K_0} is closed. Any geodesic $\widetilde{\gamma}$ realizes the strong minimum for length of curves between any two points on $\widetilde{\gamma}$, if they are sufficiently close to each other [92]. Since the group K_0 conserves the geodesic γ, it means that γ_{K_0} is an open subset of γ. Thus $\gamma_{K_0} = \gamma$ that completes the proof. □

Since the space Q is symmetric, in the algebra $\mathfrak{g}$ there exists a complementary subspace $\mathfrak{p}$ with respect to the subalgebra $\mathfrak{k}$ such that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p},\ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The space $\mathfrak{p}$ can be naturally identified with the space $T_{x_0}Q$. Under this identification the restriction of the Killing form for the algebra $\mathfrak{g}$ onto the space $\mathfrak{p}$ and the scalar product on $T_{x_0}Q$ are proportional. In particular, the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is uniquely determined by the point x_0. Let $\langle \cdot, \cdot \rangle$ be the scalar product on the algebra $\mathfrak{g}$ such that it is proportional to the Killing form and its restriction onto the subspace $\mathfrak{p} \cong T_{x_0}Q$ coincides with the restriction of the Riemannian metric g on $T_{x_0}Q$. The inclusions

$$[\mathfrak{p}, [\mathfrak{k}, \mathfrak{p}]] \subset \mathfrak{k},\ [\mathfrak{p}, [\mathfrak{k}, \mathfrak{k}]] \subset \mathfrak{p}$$

and the definition of the Killing form imply that the spaces $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal to each other with respect to the scalar product $\langle \cdot, \cdot \rangle$. From the results of [65, 111] we can extract the following key proposition.

Proposition 1.2. *The algebra* $\mathfrak{g}$ *admits the following expansion into the direct sum of subspaces:*

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \tag{1.2}$$

such that $\dim \mathfrak{a} = 1$, λ *is a nontrivial linear form on the space* $\mathfrak{a}$, $\dim \mathfrak{k}_\lambda = \dim \mathfrak{p}_\lambda = q_1$, $\dim \mathfrak{k}_{2\lambda} = \dim \mathfrak{p}_{2\lambda} = q_2$, $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$, $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$; *here* $q_1, q_2 \in \{0\} \cup \mathbb{N}$, *the subalgebra* $\mathfrak{a}$ *is the maximal commutative subalgebra in the subspace* $\mathfrak{p}$ *and it corresponds to the tangent vectors to the geodesic* $\widetilde{\gamma}$ *at the point* x_0. *All summand in* (1.2) *are* $\mathrm{ad}_{\mathfrak{k}_0}$*-invariant and the following inclusions are valid:*

$$\begin{gathered}
[\mathfrak{a}, \mathfrak{p}_\lambda] \subset \mathfrak{k}_\lambda,\ [\mathfrak{a}, \mathfrak{k}_\lambda] \subset \mathfrak{p}_\lambda,\ [\mathfrak{a}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_{2\lambda},\ [\mathfrak{a}, \mathfrak{k}_{2\lambda}] \subset \mathfrak{p}_{2\lambda},\ [\mathfrak{a}, \mathfrak{k}_0] = 0\ , \\
[\mathfrak{k}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{p}_{2\lambda} \oplus \mathfrak{a}, [\mathfrak{k}_\lambda, \mathfrak{k}_\lambda] \subset \mathfrak{k}_{2\lambda} \oplus \mathfrak{k}_0,\ [\mathfrak{p}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{k}_{2\lambda} \oplus \mathfrak{k}_0\ , \\
[\mathfrak{k}_{2\lambda}, \mathfrak{k}_{2\lambda}] \subset \mathfrak{k}_0,\ [\mathfrak{p}_{2\lambda}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_0,\ [\mathfrak{k}_{2\lambda}, \mathfrak{p}_{2\lambda}] \subset \mathfrak{a},\ [\mathfrak{k}_\lambda, \mathfrak{k}_{2\lambda}] \subset \mathfrak{k}_\lambda,\ [\mathfrak{k}_\lambda, \mathfrak{p}_{2\lambda}] \subset \mathfrak{p}_\lambda, \\
[\mathfrak{p}_\lambda, \mathfrak{k}_{2\lambda}] \subset \mathfrak{p}_\lambda,\ [\mathfrak{p}_\lambda, \mathfrak{p}_{2\lambda}] \subset \mathfrak{k}_\lambda\ .
\end{gathered} \tag{1.3}$$

Moreover, for any basis $e_{\lambda,i},\ i = 1, \ldots, q_1$ *in the space* $\mathfrak{p}_\lambda$ *and any basis* $e_{2\lambda,i},\ i = 1, \ldots, q_2$ *in the space* $\mathfrak{p}_{2\lambda}$ *there are the basis* $f_{\lambda,i},\ i = 1, \ldots, q_1$ *in the space* $\mathfrak{k}_\lambda$ *and the basis* $f_{2\lambda,i},\ i = 1, \ldots, q_2$ *in the space* $\mathfrak{k}_{2\lambda}$ *such that:*

$$\begin{gathered}
[Z, e_{\lambda,i}] = -\lambda(Z) f_{\lambda,i},\ [Z, f_{\lambda,i}] = \lambda(Z) e_{\lambda,i}\ ,\ i = 1, \ldots, q_1\ , \\
[Z, e_{2\lambda,i}] = -2\lambda(Z) f_{2\lambda,i},\ [Z, f_{2\lambda,i}] = 2\lambda(Z) e_{2\lambda,i}\ ,\ i = 1, \ldots, q_2, \forall Z \in \mathfrak{a}\ .
\end{gathered} \tag{1.4}$$

If a vector $\Lambda \in \mathfrak{a}$ *satisfies the condition* $\langle \Lambda, \Lambda \rangle = R^2$, *then* $|\lambda(\Lambda)| = \dfrac{1}{2}$.

Nonnegative integers q_1 and q_2 are said to be *multiplicities of the space* Q. The triple (q_1, q_2, R) characterize Q uniquely up to the exchange $\mathbf{S}^n \leftrightarrow \mathbf{P}^n(\mathbb{R})$. For the spaces $\mathbf{S}^n$ and $\mathbf{P}^n(\mathbb{R})$ we have $q_1 = 0,\ q_2 = n-1$; for the space $\mathbf{P}^n(\mathbb{C}):\ q_1 = 2n-2,\ q_2 = 1$; for the space $\mathbf{P}^n(\mathbb{H}):\ q_1 = 4n-4,\ q_2 = 3$; and for the space $\mathbf{P}^2(\mathbb{C}a):\ q_1 = 8,\ q_2 = 7$. Conversely, for the spaces $\mathbf{S}^n$ and $\mathbf{P}^n(\mathbb{R})$ we could reckon that $q_1 = n-1, q_2 = 0$. Our choice corresponds to the isometries $\mathbf{P}^1(\mathbb{C}) \cong \mathbf{S}^2$, $\mathbf{P}^1(\mathbb{H}) \cong \mathbf{S}^4$.

Remark 1.1. *The space* $\mathfrak{a} \oplus \mathfrak{p}_{2\lambda}$ *generates in the space* Q *a completely geodesic submanifold of the constant sectional curvature* R^{-2} *and the dimension* $q_2 + 1$. *For spaces* $\mathbf{S}^n, \mathbf{P}^n(\mathbb{C}), \mathbf{P}^n(\mathbb{H}), \mathbf{P}^2(\mathbb{C}a)$ *this submanifold is a sphere. For the space* $\mathbf{P}^n(\mathbb{R})$ *this submanifold is the whole* $\mathbf{P}^n(\mathbb{R})$. *If* $q_1 \neq 0$, *the element* Λ *and an arbitrary nonzero element from the space* $\mathfrak{p}_\lambda$ *generate in* Q *a completely geodesic two dimensional submanifolds of the constant curvature* $(2R)^{-2}$.

Let vectors $e_{\lambda,i}, f_{\lambda,i},\ i = 1, \ldots, q_1$ and $e_{2\lambda,i}, f_{2\lambda,i},\ i = 1, \ldots, q_2$ be as in Proposition 1.2. Choose a vector $\Lambda \in \mathfrak{a}$ such that $\lambda(\Lambda) = \frac{1}{2}$. The following proposition easily follows from Proposition 1.2.

Proposition 1.3. *The spaces* $\mathfrak{a} \oplus \mathfrak{k}_0$, $\mathfrak{k}_\lambda \oplus \mathfrak{p}_\lambda$, $\mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_{2\lambda}$ *are pairwise orthogonal. One has*

$$\begin{gathered}
\langle e_{\lambda,i}, e_{\lambda,j} \rangle = \langle f_{\lambda,i}, f_{\lambda,j} \rangle,\ \langle e_{\lambda,i}, f_{\lambda,j} \rangle = -\langle f_{\lambda,i}, e_{\lambda,j} \rangle = 0,\ i,j = 1, \ldots, q_1\ , \\
\langle e_{2\lambda,i}, e_{2\lambda,j} \rangle = \langle f_{2\lambda,i}, f_{2\lambda,j} \rangle, \langle e_{2\lambda,i}, f_{2\lambda,j} \rangle = -\langle f_{2\lambda,i}, e_{2\lambda,j} \rangle = 0,\ i,j = 1, \ldots, q_2\ .
\end{gathered} \tag{1.5}$$

Proof. The Ad_G-invariance of the metric $\langle\cdot,\cdot\rangle$ implies that the operator $T_\Lambda : X \to [\Lambda,[\Lambda,X]]$ is symmetric on the space $\mathfrak{g}$. This operator has the following eigenspaces $\mathfrak{a}\oplus\mathfrak{k}_0,\ \mathfrak{k}_\lambda\oplus\mathfrak{p}_\lambda,\ \mathfrak{k}_{2\lambda}\oplus\mathfrak{p}_{2\lambda}$ with eigenvalues $0,\ -\lambda^2(\Lambda) = -\frac{1}{4}, -4\lambda^2(\Lambda) = -1$, respectively. Thus, these eigenspaces are orthogonal to each other. The Ad_G-invariance of the metric $\langle\cdot,\cdot\rangle$ and the equality (1.4) give

$$\lambda(\Lambda)\langle e_{\lambda,i}, e_{\lambda,j}\rangle = \langle[\Lambda, f_{\lambda,i}], e_{\lambda,j}\rangle = -\langle f_{\lambda,i}, [\Lambda, e_{\lambda,j}]\rangle = \lambda(\Lambda)\langle f_{\lambda,i}, f_{\lambda,j}\rangle \ .$$

This proves the first equality from (1.5). The orthogonality $\mathfrak{p}\perp\mathfrak{k}$ implies the second and the forth equalities from (1.5). The third equality is similar to the first. □

The Jacobi identity and formulae (1.4) give $[Z,[e_{\lambda,i}, f_{\lambda,i}]] = 0$. Therefore, the relation $[\mathfrak{k}_\lambda,\mathfrak{p}_\lambda] \subset \mathfrak{p}_{2\lambda}\oplus\mathfrak{a}$ from (1.3) implies $[e_{\lambda,i}, f_{\lambda,i}] \in \mathfrak{a}$. Let $[e_{\lambda,i}, f_{\lambda,i}] =: \varkappa_i\Lambda$. The Ad_G-invariance of the metric $\langle\cdot,\cdot\rangle$ leads to

$$0 = \langle\Lambda,[e_{\lambda,i}, f_{\lambda,i}]\rangle + \langle[e_{\lambda,i},\Lambda], f_{\lambda,i}\rangle = \varkappa_i\langle\Lambda,\Lambda\rangle + \lambda(\Lambda)\langle f_{\lambda,i}, f_{\lambda,i}\rangle \ ,$$

and using the first equality from (1.5) we obtain:

$$\varkappa_i = -\frac{\lambda(\Lambda)}{\langle\Lambda,\Lambda\rangle}\langle f_{\lambda,i}, f_{\lambda,i}\rangle = -\frac{\lambda(\Lambda)}{\langle\Lambda,\Lambda\rangle}\langle e_{\lambda,i}, e_{\lambda,i}\rangle \ .$$

Similarly, we get:

$$[e_{2\lambda,i}, f_{2\lambda,i}] = -\frac{2\lambda(\Lambda)}{\langle\Lambda,\Lambda\rangle}\langle f_{2\lambda,i}, f_{2\lambda,i}\rangle\Lambda = -\frac{2\lambda(\Lambda)}{\langle\Lambda,\Lambda\rangle}\langle e_{2\lambda,i}, e_{2\lambda,i}\rangle\Lambda \ .$$

Using the freedom provided by Proposition 1.2, we choose the bases $\{e_{\lambda,i}\}_{i=1}^{q_1}$ in the space $\mathfrak{p}_\lambda$ and $\{e_{2\lambda,j}\}_{j=1}^{q_2}$ in the space $\mathfrak{p}_{2\lambda}$ to be orthogonal, with norms of all their elements equal R. Thus, the elements $\Lambda, e_{\lambda,i}, e_{2\lambda,j}\ , i = 1,\ldots,q_1, j = 1,\ldots,q_2$ form the orthogonal basis in the space $\mathfrak{p}$ and the elements $f_{\lambda,i},\ \ f_{2\lambda,j}, i = 1,\ldots,q_1, j = 1,\ldots,q_2$ form the orthogonal basis in the space $\mathfrak{k}_\lambda\oplus\mathfrak{k}_{2\lambda}$, due to Proposition 1.3. Note that due to (1.5) $\langle f_{\lambda,i}, f_{\lambda,i}\rangle = R^2, i = 1,\ldots,q_1,\ \langle f_{2\lambda,j}, f_{2\lambda,j}\rangle = R^2, j = 1,\ldots,q_2$.

Proposition 1.4. *1. The relations* (1.4) *can be rewritten in the following form:*

$$\begin{aligned}
&[\Lambda, e_{\lambda,i}] = -\frac{1}{2}f_{\lambda,i},\ [\Lambda, f_{\lambda,i}] = \frac{1}{2}e_{\lambda,i},\ [e_{\lambda,i}, f_{\lambda,i}] = -\frac{1}{2}\Lambda\ ,\\
&\langle e_{\lambda,i}, e_{\lambda,j}\rangle = \langle f_{\lambda,i}, f_{\lambda,j}\rangle = \delta_{ij}R^2,\ i,j = 1,\ldots,q_1\ ,\\
&[\Lambda, e_{2\lambda,i}] = -f_{2\lambda,i},\ [\Lambda, f_{2\lambda,i}] = e_{2\lambda,i},\ [e_{2\lambda,i}, f_{2\lambda,i}] = -\Lambda\ ,\\
&\langle e_{2\lambda,i}, e_{2\lambda,j}\rangle = \langle f_{2\lambda,i}, f_{2\lambda,j}\rangle = \delta_{ij}R^2,\ i,j = 1,\ldots,q_2,\ \langle\Lambda,\Lambda\rangle = R^2.
\end{aligned} \tag{1.6}$$

2. Let X and Y be some elements from the basis

$$\Lambda, e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,j}, f_{2\lambda,j},\ i = 1,\ldots,q_1, j = 1,\ldots,q_2 \tag{1.7}$$

of the space $\mathfrak{m} := \mathfrak{a}\oplus\mathfrak{k}_\lambda\oplus\mathfrak{k}_{2\lambda}\oplus\mathfrak{p}_\lambda\oplus\mathfrak{p}_{2\lambda}$. Let $X'_\mathfrak{m}$ be a projection of an element $X' \in \mathfrak{g}$ onto the space $\mathfrak{m}$ with respect to the expansion $\mathfrak{g} = \mathfrak{k}_0\oplus\mathfrak{m}$. Expand the element $[X,Y]_\mathfrak{m}$ in the basis (1.7)*. Then its coordinates with respect to the elements X, Y are equal to zero.*

Proof. Relations (1.6) are evident. In view of the inclusions from Proposition 1.2 it is sufficient to prove the second statement only in the following cases: a) $X = e_{\lambda,i}$, $Y = f_{2\lambda,j}$ and b) $X = f_{\lambda,i}$, $Y = f_{2\lambda,j}$. Consider the case a). From (1.3) we get $[f_{2\lambda,j}, e_{\lambda,i}] \in \mathfrak{p}_\lambda$. The Ad_G-invariance of the metric $\langle\cdot,\cdot\rangle$ gives $\langle [f_{2\lambda,j}, e_{\lambda,i}], e_{\lambda,i}\rangle = -\langle e_{\lambda,i}, [f_{2\lambda,j}, e_{\lambda,i}]\rangle$, $i = 1,\ldots,q_1, j = 1,\ldots,q_2$ and then $[f_{2\lambda,j}, e_{\lambda,i}] \perp e_{\lambda,i}$. Now, taking into account the orthogonality of the basis $\{e_{\lambda,i}\}_{i=1}^{q_1}$ of the space $\mathfrak{p}_\lambda$, we obtain the second statement in the case a). The case b) is completely similar. □

In the following proposition the useful information on Lie algebras $\mathfrak{g}$, corresponding to curved two-point homogeneous Riemannian spaces is collected. This information can be found in [28, 65, 134, 135].

Proposition 1.5. *Noncompact two-point homogeneous spaces of types 7,8,9,10 are analogous to the compact two-point homogeneous spaces of types 2(3),4,5,6, respectively. In particular, it means that Lie algebras $\mathfrak{g}$ of symmetry groups of analogous spaces are different real forms of a simple (except cases $\mathbf{S}^3, \mathbf{P}^3(\mathbb{R})$, $\mathbf{H}^3(\mathbb{R})$, see below) complex Lie algebra $\mathfrak{g}(\mathbb{C})$. The transition from one such real form to another one can be done by multiplying the subspace $\mathfrak{p}$ from the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ by the imaginary unit $\mathbf{i}$ (or by $-\mathbf{i}$).*[2]

For spaces $\mathbf{S}^{2n}, \mathbf{P}^{2n}(\mathbb{R}), \mathbf{H}^{2n}(\mathbb{R})$, $n \geqslant 1$ one has $\mathfrak{g}(\mathbb{C}) = \mathfrak{so}(2n+1,\mathbb{C}) = \mathfrak{B}_n$; for spaces $\mathbf{S}^{2n-1}$, $\mathbf{P}^{2n-1}(\mathbb{R}), \mathbf{H}^{2n-1}(\mathbb{R})$, $n \geqslant 3$: $\mathfrak{g}(\mathbb{C}) = \mathfrak{so}(2n,\mathbb{C}) = \mathfrak{D}_n$; for spaces $\mathbf{P}^n(\mathbb{C}), \mathbf{H}^n(\mathbb{C})$, $n \geqslant 2$: $\mathfrak{g}(\mathbb{C}) = \mathfrak{sl}(n+1,\mathbb{C}) = \mathfrak{A}_n$; for spaces $\mathbf{P}^n(\mathbb{H}), \mathbf{H}^n(\mathbb{H})$, $n \geqslant 2$: $\mathfrak{g}(\mathbb{C}) = \mathfrak{sp}(2(n+1),\mathbb{C}) = \mathfrak{C}_{n+1}$; for spaces $\mathbf{P}^2(\mathbb{C}a), \mathbf{H}^2(\mathbb{C}a)$: $\mathfrak{g}(\mathbb{C}) = \mathfrak{f}_4$.

Here $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n, \mathfrak{D}_n$ are four series of simple complex Lie algebras of the rank $n \in \mathbb{N}$ and the Lie algebra $\mathfrak{f}_4$ of the rank 4 is one of the five simple exceptional complex Lie algebras[3]*. Their real forms have the same rank n. For the spaces $\mathbf{S}^3, \mathbf{P}^3(\mathbb{R}), \mathbf{H}^3(\mathbb{R})$ one has the complex Lie algebra $\mathfrak{g}(\mathbb{C}) = \mathfrak{so}(4,\mathbb{C}) = \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$ (the direct sum of Lie algebras). Its real form $\mathfrak{so}(1,3)$, corresponding to the space $\mathbf{H}^3(\mathbb{R})$, is simple and its compact real form, corresponding to the spaces $\mathbf{S}^3(\mathbb{R}), \mathbf{P}^3(\mathbb{R})$, is not: $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.*

Let $S(\mathfrak{g})^G \equiv S(\mathfrak{g})^I$ be the subalgebra of Ad_G-invariant elements in the commutative symmetric algebra $S(\mathfrak{g})$ for the space $\mathfrak{g}$ (see Sect. 2.1.2 *below). The algebra $S(\mathfrak{A}_n)^I$ is freely generated by polynomials of degrees:* $2,3,4\ldots n+1$*; the algebras $S(\mathfrak{B}_n)^I$ and $S(\mathfrak{C}_n)^I$ by polynomials of degrees:* $2,4,6\ldots 2n$*; the algebra $S(\mathfrak{D}_n)^I$ by polynomials of degrees:* $2,4,6\ldots 2n-2, n$*; the algebra $S(\mathfrak{f}_4)^I$ by polynomials of degrees:* $2,6,8,12$.

Due to the natural isomorphism $(\mathfrak{g}^*)^* \cong \mathfrak{g}$ the algebra $S(\mathfrak{g})$ is isomorphic to the algebra $\mathcal{P}(\mathfrak{g}^*)$ of all polynomial functions on the dual space $\mathfrak{g}^*$. Thus, the previous paragraph gives also degrees of independent generators for algebras of invariants of Ad_G^*-action, where G runs over simple complex Lie groups or their real forms.

[2] Actually this transformation is the Weyl unitary trick from representation theory.

[3] Other exceptional complex simple Lie algebras are $\mathfrak{g}_2, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, where indices are their ranks. Their compact real forms are denoted by the same way.

A subalgebra $\mathfrak{a}'$ of a real semisimple Lie algebra $\mathfrak{g}'$ is called $\mathbb{R}$-*diagonalizable* if all operators ad_x, $x \in \mathfrak{a}'$ correspond to diagonal matrices with respect to some base in $\mathfrak{g}'$. A *real rank* $\mathrm{rk}_{\mathbb{R}}\, \mathfrak{g}'$ of $\mathfrak{g}'$ is the dimension of any its maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}' \in \mathfrak{g}'$ [65, 135]. The real rank of a compact semisimple Lie algebra is null, in particular for the algebras $\mathfrak{g}$ of local isometries, corresponding to the spaces of types 2–6. The real rank of $\mathfrak{g}$ for noncompact curved two-point homogeneous Riemannian spaces equals 1 and $\mathbf{i}\mathfrak{a}$ is one of the corresponding diagonalizable subalgebras, where $\mathfrak{a}$ is given in Proposition 1.2.

It is known that every normal discreet subgroup of any connected topological group G' lies in the center of G' ([142], lecture 9) and for every connected Lie group G there is its universal covering $\tilde{\pi} : \widetilde{G} \to G$ by connected and simply connected Lie group $\widetilde{G}$. Therefore, the kernel $\tilde{\pi}^{-1}(e)$ of this covering lies in the center of $\widetilde{G}$ and the adjoint (coadjoint) orbits of the groups $\widetilde{G}$ and $G \cong \widetilde{G}/\tilde{\pi}^{-1}(e)$ are the same. Let us summarize this fact in the proposition, useful in the followings.

Proposition 1.6. *Let G' and G'' be connected locally isomorphic Lie groups. Then $\mathrm{Ad}_{G'}$-orbits coincide with $\mathrm{Ad}_{G''}$-orbits and $\mathrm{Ad}^*_{G'}$-orbits coincide with $\mathrm{Ad}^*_{G''}$-orbits. If G_1 is a Lie group (not necessarily connected) locally isomorphic to G', then each Ad_{G_1}-orbit consist of some (may be one) $\mathrm{Ad}_{G'}$-orbits, diffeomorphic with each other. The same is valid also for $\mathrm{Ad}^*_{G_1}$-orbits.*

1.3 Models of Classical Compact Two-Point Homogeneous Riemannian Spaces

1.3.1 The Model for the Space $\mathbf{P}^n(\mathbb{H})$

Let $\mathbb{H}$ be the quaternion algebra over the field $\mathbb{R}$ with the base $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$, where $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The *quaternion conjugation* acts as follows: $\overline{x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k}} = x - y\mathbf{i} - z\mathbf{j} - t\mathbf{k}$, $x, y, z, t \in \mathbb{R}$.

Let $\mathbb{H}^{n+1}$ be the right quaternion space and $(z_1, \ldots, z_{n+1})$ be coordinates on it. Let *quaternion projective space* $\mathbf{P}^n(\mathbb{H})$ be a factor space of the space $\mathbb{H}^{n+1}\backslash(0)$ with respect to the right action of the multiplicative group $\mathbb{H}^* = \mathbb{H}\backslash(0)$. The set $(z_1 : \ldots : z_{n+1})$ up to the multiplication from the right by an arbitrary element from the group $\mathbb{H}^*$ is the set of homogeneous coordinates for the element[4] $\pi_1(\mathbf{z})$ on the space $\mathbf{P}^n(\mathbb{H})$, where $\pi_1 : \mathbb{H}^{n+1}\backslash(0) \to \mathbf{P}^n(\mathbb{H})$ is the canonical projection. Let $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n+1} \bar{x}_i y_i$, $\mathbf{x} = (x_1, \ldots, x_{n+1})$, $\mathbf{y} = (y_1, \ldots, y_{n+1}) \in \mathbb{H}^{n+1}$ be the standard scalar product in the space $\mathbb{H}^{n+1}$. Let $\mathbf{z} \in \mathbb{H}^{n+1}\backslash(0)$, $\xi_i \in T_{\mathbf{z}}\mathbb{H}^{n+1}$, $\zeta_i = \pi_{1*}\xi_i \in T_{\pi_1(\mathbf{z})}\left(\mathbf{P}^n(\mathbb{H})\right)$, $i = 1, 2$. A metric

$$\tilde{g}|_{\mathbf{z}}(\zeta_1, \zeta_2) = \left(\langle \xi_1, \xi_2 \rangle \langle \mathbf{z}, \mathbf{z} \rangle - \langle \xi_1, \mathbf{z} \rangle \langle \mathbf{z}, \xi_2 \rangle\right) / \langle \mathbf{z}, \mathbf{z} \rangle^2 \,, \tag{1.8}$$

on the space $\mathbf{P}^n(\mathbb{H})$ is the analogue for the metric with a constant sectional curvature on the space $\mathbf{P}^n(\mathbb{R})$ (see Sect. 1.3.3) and the metric with a constant

[4] To distinguish the point $\mathbf{x} \in M$ from their coordinates we shall single out it by the bold type.

holomorphic sectional curvature on the space $\mathbf{P}^n(\mathbb{C})$ (see Sect. 1.3.2). The real part of the metric (1.8) is a Riemannian metric on the space $\mathbf{P}^n(\mathbb{H})$:

$$g = 4R^2 \operatorname{Re} \tilde{g} \,. \tag{1.9}$$

The normalizing factor $4R^2$ in (1.9) is chosen due to the following reasons. The space $\mathbf{P}^1(\mathbb{H})$ with this metric is the sphere $\mathbf{S}^4$ with the standard metric of the constant sectional curvature R^{-2}. To see this one can consider a homeomorphism $\nu : \mathbf{P}^1(\mathbb{H}) \to \overline{\mathbb{H}} \cong \mathbf{S}^4$, $\nu(z_1, z_2) = z_1 \left(z_2\right)^{-1} = z \in \overline{\mathbb{H}}$, where $\overline{\mathbb{H}}$ is the quaternion space completed with the point at infinity. For $n = 1$ formula (1.9) has the form

$$g = 4R^2 \frac{(d\bar{z}_1 dz_1 + d\bar{z}_2 dz_2)(|z_1|^2 + |z_2|^2) - (d\bar{z}_1 \cdot z_1 + d\bar{z}_2 \cdot z_2)(\bar{z}_1 dz_1 + \bar{z}_2 dz_2)}{(|z_1|^2 + |z_2|^2)^2}. \tag{1.10}$$

Using the formula $|z_2|^2 dz_1 - z_1 \bar{z}_2 dz_2 = |z_2|^2 (dz) z_2$ by direct calculations one can reduce expression (1.10) to the form:

$$g = \frac{4R^2 dz d\bar{z}}{\left(1 + |z|^2\right)^2} \,,$$

which is the metric of the constant sectional curvature R^{-2} on the sphere $\mathbf{S}^4$ (see Sect. 1.3.3 below).

Let $B_M(\rho)$ be a geodesic ball of the radius ρ, $\mathbf{S}_M(\rho)$ be its boundary (a geodesic sphere of the radius ρ) in a Riemannian space M, and vol$(\cdot)$ be the volume function, generated by the metric. It is known [55, 66] that

$$\operatorname{vol}\left(B_{\mathbf{P}^n(\mathbb{H})}(\rho)\right) = \frac{\pi^{2n}(2R)^{4n}}{(2n+1)!} \sin^{4n}\left(\frac{\rho}{2R}\right)\left(1 + 2n\cos^2\left(\frac{\rho}{2R}\right)\right).$$

This implies

$$\operatorname{vol}\left(\mathbf{S}_{\mathbf{P}^n(\mathbb{H})}(\rho)\right) = \frac{d}{d\rho}\operatorname{vol}\left(B_{\mathbf{P}^n(\mathbb{H})}(\rho)\right) = \frac{\pi^{2n}(2R)^{4n-1}}{4(2n-1)!} \sin^{4(n-1)}\left(\frac{\rho}{2R}\right)\sin^3\left(\frac{\rho}{R}\right) \,,$$
$$\operatorname{vol}\left(\mathbf{P}^n(\mathbb{H})\right) = \operatorname{vol}\left(B_{\mathbf{P}^n(\mathbb{H})}(\pi R)\right) = \frac{\pi^{2n}(2R)^{4n}}{(2n+1)!} \,.$$

The left action of the group $\mathrm{U}_{\mathbb{H}}(n+1)$, consisting of quaternion matrices A of the size $(n+1)\times(n+1)$ such that $\bar{A}^T A = E$, conserves the scalar product $\langle\cdot,\cdot\rangle$ in the space $\mathbb{H}^{n+1}$. The real dimension of this group is $(2n+3)(n+1)$. If we write quaternion coordinates in $\mathbb{H}^{n+1}$ as pairs of complex numbers, then the group $\mathrm{U}_{\mathbb{H}}(n+1)$ becomes the symplectic group $\mathrm{Sp}(n+1)$.

Left and right multiplications evidently commute, so the left action of the group $\mathrm{U}_{\mathbb{H}}(n+1)$ is correctly defined also on the space $\mathbf{P}^n(\mathbb{H})$. Obviously, it is transitive and conserves the metric g. The stationary subgroup of the point from the space $\mathbf{P}^n(\mathbb{H})$ with the homogeneous coordinates $(1, 0, \ldots, 0)$ is the group $\mathrm{U}_{\mathbb{H}}(n)\,\mathrm{U}_{\mathbb{H}}(1)$, where the group $\mathrm{U}_{\mathbb{H}}(n)$ acts onto the last n coordinates, and the group $\mathrm{U}_{\mathbb{H}}(1)$ acts by the left multiplication of all homogeneous coordinates by quaternions with the unit norm. Stationary subgroups for all points of a homogeneous space are conjugated and hence isomorphic. This yields the following isomorphism

$$\mathbf{P}^n(\mathbb{H}) \simeq \mathrm{U}_{\mathbb{H}}(n+1)/(\mathrm{U}_{\mathbb{H}}(n)\,\mathrm{U}_{\mathbb{H}}(1)) \ .$$

The Lie algebra $\mathfrak{u}_{\mathbb{H}}(n+1)$ consists of quaternion matrices A of the size $(n+1)\times(n+1)$ such that $\bar{A}^T = -A$. Let E_{kj} be the matrix of the size $(n+1)\times(n+1)$ with the unique nonzero element equals 1, locating at the intersection of the kth row and the jth column. Choose the base for the algebra $\mathfrak{u}_{\mathbb{H}}(n+1)$ as:

$$\begin{aligned} &\Psi_{kj} = \frac{1}{2}(E_{kj} - E_{jk}),\ 1 \leqslant k < j \leqslant n+1,\ \Upsilon_{kj} = \frac{\mathbf{i}}{2}(E_{kj} + E_{jk}) \ , \\ &\Omega_{kj} = \frac{\mathbf{j}}{2}(E_{kj} + E_{jk}),\ \Theta_{kj} = \frac{\mathbf{k}}{2}(E_{kj} + E_{jk}),\ 1 \leqslant k \leqslant j \leqslant n+1 \ . \end{aligned} \tag{1.11}$$

The commutator relations for these elements are:

$$\begin{aligned} [\Psi_{kj}, \Psi_{ml}] &= \frac{1}{2}\left(\delta_{jm}\Psi_{kl} - \delta_{km}\Psi_{jl} + \delta_{kl}\Psi_{jm} - \delta_{jl}\Psi_{km}\right) \ , \\ [\Psi_{kj}, \Upsilon_{ml}] &= \frac{1}{2}\left(\delta_{jm}\Upsilon_{kl} - \delta_{km}\Upsilon_{jl} + \delta_{lj}\Upsilon_{km} - \delta_{lk}\Upsilon_{jm}\right) \ , \\ [\Upsilon_{kj}, \Upsilon_{ml}] &= \frac{1}{2}\left(\delta_{jm}\Psi_{lk} + \delta_{km}\Psi_{lj} + \delta_{kl}\Psi_{mj} + \delta_{jl}\Psi_{mk}\right) \ , \\ [\Upsilon_{kj}, \Omega_{ml}] &= \frac{1}{2}\left(\delta_{jm}\Theta_{lk} + \delta_{km}\Theta_{lj} + \delta_{kl}\Theta_{mj} + \delta_{jl}\Theta_{mk}\right) \ , \end{aligned} \tag{1.12}$$

plus the analogous equalities, obtaining from the latter three relations by the cyclic permutation $\Upsilon \to \Omega \to \Theta \to \Upsilon$, where $\Psi_{kj} = -\Psi_{jk}$, $\Psi_{kk} = 0$, $\Upsilon_{kj} = \Upsilon_{jk}$, $\Omega_{kj} = \Omega_{jk}$, $\Theta_{kj} = \Theta_{jk}$.

1.3.2 The Model for the Space $\mathbf{P}^n(\mathbb{C})$

Taking the factor space of $\mathbb{C}^{n+1}\backslash(0)$ w.r.t the action of the multiplicative group $\mathbb{C}^* = \mathbb{C}\backslash(0)$ (due to the commutativity of the complex multiplication it makes no difference left or right), one gets *the complex projective space* $\mathbf{P}^n(\mathbb{C})$. Let $\pi_1 : \mathbb{C}^{n+1}\backslash(0) \to \mathbf{P}^n(\mathbb{C})$ be the canonical projection. Let now $\langle \mathbf{x}, \mathbf{y}\rangle := \sum_{i=1}^{n+1} \bar{x}_i y_i$, $\mathbf{x} = (x_1, \ldots, x_{n+1})$, $\mathbf{y} = (y_1, \ldots, y_{n+1}) \in \mathbb{C}^{n+1}$ be the standard scalar product in the space $\mathbb{C}^{n+1}$.

The metric $\tilde{g}$ of the constant holomorphic sectional curvature on the space $\mathbf{P}^n(\mathbb{C})$ is defined by the same formula (1.8) as on the space $\mathbf{P}^n(\mathbb{H})$, where now $\mathbf{z} \in \mathbb{C}^{n+1}\backslash(0)$, $\xi_i \in T_{\mathbf{z}}\mathbb{C}^{n+1}$, $\zeta_i = \pi_{1*}\xi_i \in T_{\pi_1(\mathbf{z})}\left(\mathbf{P}^n(\mathbb{C})\right)$, $i = 1, 2$.

The Riemannian metric g on the space $\mathbf{P}^n(\mathbb{C})$ is:

$$g = 4R^2 \operatorname{Re}\tilde{g} \ . \tag{1.13}$$

For $n = 2$ it is not difficult to verify (like in Sect. 1.3.1) that the homeomorphism $\tau : \mathbf{P}^1(\mathbb{C}) \to \bar{\mathbb{C}} \cong \mathbf{S}^2$, $\tau(z_1, z_2) = z_1 (z_2)^{-1} = z \in \overline{\mathbb{C}}$, transforms (1.13) into the metric

$$g = \frac{4R^2 dz d\bar{z}}{(1+|z|^2)^2} ,$$

of the sectional curvature R^{-2} on the sphere $\mathbf{S}^2$ (see Sect. 1.3.3 below).

For the complex projective space one has [55, 66]

$$\begin{aligned} \operatorname{vol}\left(B_{\mathbf{P}^n(\mathbb{C})}(\rho)\right) &= \frac{\pi^n (2R)^{2n}}{n!} \sin^{2n}\left(\frac{\rho}{2R}\right), \; 0 \leqslant \rho \leqslant \pi R , \\ \operatorname{vol}\left(\mathbf{S}_{\mathbf{P}^n(\mathbb{C})}(\rho)\right) &= \frac{d}{d\rho} \operatorname{vol}\left(B_{\mathbf{P}^n(\mathbb{C})}(\rho)\right) = \frac{\pi^n (2R)^{2n-1}}{(n-1)!} \sin^{2(n-1)}\left(\frac{\rho}{2R}\right) \sin\left(\frac{\rho}{R}\right) , \\ \operatorname{vol}\left(\mathbf{P}^n(\mathbb{C})\right) &= \operatorname{vol}\left(B_{\mathbf{P}^n(\mathbb{C})}(\pi R)\right) = \frac{(4\pi R^2)^n}{n!} . \end{aligned}$$

The left action of the group $G = \mathrm{SU}(n+1)$ on the space $\mathbb{C}^{n+1}$ conserves the scalar product $\langle \cdot, \cdot \rangle$ and induces the action of $\mathrm{SU}(n+1)$ in the space $\mathbf{P}^n(\mathbb{C})$, conserving metrics $\tilde{g}$ and g.

The stationary subgroup, corresponding to the point of the space $\mathbf{P}^n(\mathbb{C})$ with homogeneous coordinates $(1 : 0 : \ldots : 0)$, is the group $\mathrm{U}(n) = \mathrm{SU}(n)\,\mathrm{U}(1)$, where the factor $\mathrm{SU}(n)$ acts in the standard way onto the last n coordinates, and the factor $\mathrm{U}(1)$ acts by the multiplication of the first coordinate by $e^{\mathbf{i}\phi}$ and the second one by $e^{-\mathbf{i}\phi}$, $\phi \in \mathbb{R} \mod 2\pi$. This leads to the isomorphism $\mathbf{P}^n(\mathbb{C}) \simeq \mathrm{SU}(n+1)/\,\mathrm{U}(n)$.

Choose a base of the algebra $\mathfrak{su}(n+1)$ in the form:

$$\begin{aligned} \Psi_{kj} &= \frac{1}{2}(E_{kj} - E_{jk}),\; \Upsilon_{kj} = \frac{\mathbf{i}}{2}(E_{kj} + E_{jk}), 1 \leqslant k < j \leqslant n+1 , \\ \Upsilon_k &= \frac{\mathbf{i}}{2}(E_{11} - E_{kk}) , \; 2 \leqslant k \leqslant n+1 . \end{aligned} \tag{1.14}$$

The commutators for these elements are easily extracted from (1.12), taking into account the equality $\Upsilon_k = \frac{1}{2}(\Upsilon_{11} - \Upsilon_{kk})$.

1.3.3 Models for Spaces $\mathbf{S}^n$, $\mathbf{P}^n(\mathbb{R})$ and $\mathbf{H}^n(\mathbb{R})$

Let now $\langle \cdot, \cdot \rangle$ be the standard scalar product in the space $\mathbb{R}^{n+1}$. The equation $\langle \mathbf{x}, \mathbf{x} \rangle = R > 0$ defines the sphere $\mathbf{S}^n \cong \mathrm{SO}(n+1)/\,\mathrm{SO}(n) \subset \mathbb{R}^{n+1}$ of the radius R with the induced metric on it. The *real projective space* $\mathbf{P}^n(\mathbb{R})$ is the factor space of $\mathbf{S}^n$ w.r.t. the relation: $\mathbf{x} \sim -\mathbf{x}$.

Let

$$\Psi_{kj} = \frac{1}{2}(E_{kj} - E_{jk}),\; 1 \leqslant k < j \leqslant n+1 \tag{1.15}$$

be the base of the algebra $\mathfrak{so}(n+1)$. The commutator relations for them are contained in (1.12).

The sphere $\mathbf{S}^n$ can be described also as the space $\mathbb{R}^n \cup \infty$ with the metric

$$g_s = 4R^2 \sum_{i=1}^{n} dy_i^2 \bigg/ \left(1 + \sum_{i=1}^{n} dy_i^2\right)^2 , \tag{1.16}$$

where y_i are Cartesian coordinates in $\mathbb{R}^n$.

Indeed, consider the stereographic projection of the sphere $\mathbf{S}^n \subset \mathbb{R}^{n+1}$ from the point $\mathbf{x}' = (0, \dots, 0, -R)$ onto the subspace $P_1 = (\mathbf{x} \in \mathbb{R}^n | \, x_{n+1} = 0)$. Let the line passing through points $\mathbf{x}, \mathbf{x}' \in \mathbf{S}^n$ intersects the subspace P_1 at the point $\tilde{\mathbf{x}}$. Then this projection maps the point $\mathbf{x}$ into the point $\tilde{\mathbf{x}}$. The point $\mathbf{x}'$ is mapped into ∞. Let $\mathcal{C}$ be the equator of $\mathbf{S}^n$ lying in the plane P_1. It consists of stationary points w.r.t. this projection. Big circles on the sphere $\mathbf{S}^n$ are geodesics and they are transformed into those Euclidean circles on P_1, that intersect $\mathcal{C}$ in antipodal points, and into Euclidean straight lines passing through $0 \in P_1$.

Simple computations give

$$\tilde{x}_i = \frac{Rx_i}{R + x_{n+1}}, \; i = 1, \dots, n, \; \tilde{r}^2 := \sum_{i=1}^{n} \tilde{x}_i^2 = \frac{R - x_{n+1}}{R + x_{n+1}} R^2 , \tag{1.17}$$

and formulas for the inverse map:

$$x_i = \frac{2R^2 \tilde{x}_i}{R^2 + \tilde{r}^2}, \; i = 1, \dots, n, \; x_{n+1} = \frac{R^2 - \tilde{r}^2}{R^2 + \tilde{r}^2} R . \tag{1.18}$$

It is easily obtained that under this map the restriction of the Euclidean metric $\sum_{i=1}^{n+1} dx_i^2$ in $\mathbb{R}^{n+1}$ onto $\mathbf{S}^n$ is transformed into the metric

$$g_s = 4R^4 \sum_{i=1}^{n} d\tilde{x}_i^2 \bigg/ \left(R^2 + \tilde{r}^2\right)^2 ,$$

and the substitution $\tilde{x}_i =: Ry_i$ leads to (1.16).

One can write the metric (1.16) in the spherical form

$$g_s = 4R^2 \left(dr^2 + r^2 \tilde{g}_s\right) / \left(1 + r^2\right)^2 , \; r^2 := \sum_{i=1}^{n} y_i^2 , \tag{1.19}$$

where $\tilde{g}_s$ is the standard metric on the unit sphere $\mathbf{S}^{n-1}$. For the new coordinate $v = 2r/(1 - r^2)$ it also holds

$$g_s = R^2 \left(\frac{dv^2}{(1 + v^2)^2} + \frac{v^2 \tilde{g}_s}{1 + v^2} \right) . \tag{1.20}$$

This form of the metric corresponds to the stereographic projection of the sphere $\mathbf{S}^n$ from the point $0 \in \mathbb{R}^{n+1}$ onto the subspace $P_2 = (\mathbf{x} \in \mathbb{R}^{n+1} | \, x_{n+1} = R)$, where vR is the Euclidean distance between a point from P_2 and the fixed point $\mathbf{x}'' = (0, \dots, 0, R) \in \mathbf{S}^n$. Obviously, this projection is a double covering of P_2 with singularities on the sphere $\mathbf{S}^{n-1} = \mathbf{S}^n \cap P_1$, which is mapped into ∞. Straight lines in P_2 are geodesics of metric (1.20) since they are images of big circles on $\mathbf{S}^n$. Let ρ be the distance from the point $r = 0$ in the metric (1.19). Then one gets

$$r = \tan\left(\frac{\rho}{2R}\right), \ v = \tan\left(\frac{\rho}{R}\right). \tag{1.21}$$

Now one has [55]:

$$\mathrm{vol}\,(B_{\mathbf{S}^n}(\rho)) = \frac{2\pi^{n/2}R^n}{\Gamma(\frac{n}{2})}\int_0^{\rho/R}\sin^{n-1}t\,dt,\ 0 \leqslant \rho \leqslant \pi R\,,$$

$$\mathrm{vol}\,(\mathbf{S}_{\mathbf{S}^n}(\rho)) = \frac{d}{d\rho}\,\mathrm{vol}\,(B_{\mathbf{S}^n}(\rho)) = \frac{2\pi^{n/2}R^{n-1}}{\Gamma(\frac{n}{2})}\sin^{n-1}\left(\frac{\rho}{R}\right)\,,$$

$$\mathrm{vol}\,(\mathbf{S}^n) = \mathrm{vol}\,(B_{\mathbf{S}^n}(\pi R)) = \frac{2\pi^{\frac{n+1}{2}}R^n}{\Gamma(\frac{n+1}{2})}\,,$$

where Γ is the gamma-function (see appendix B). The first two formulas for $\mathbf{P}^n(\mathbb{R})$ are the same with $0 \leqslant \rho \leqslant \pi R/2$ and

$$\mathrm{vol}\,(\mathbf{P}^n(\mathbb{R})) = \frac{\pi^{\frac{n+1}{2}}R^n}{\Gamma(\frac{n+1}{2})}\,.$$

Consider the similar construction for the hyperbolic space $\mathbf{H}^n(\mathbb{R})$ embedded in the standard way as a one sheet of the two-sheet hyperboloid

$$-\sum_{i=1}^n x_i^2 + x_{n+1}^2 = R^2,\ x_{n+1} \geqslant R > 0 \tag{1.22}$$

into the space $\mathbb{R}^{n+1}$ endowed with the Minkowski metric

$$\sum_{i=1}^n dx_i^2 - dx_{n+1}^2\,. \tag{1.23}$$

The stereographic projection from the point $\mathbf{x}' = (0,\dots,0,-R)$ onto the subspace P_1 maps the sheet of hyperboloid with $x_{n+1} \geqslant R > 0$ onto the open ball in P_1 of the radius R with the center at $0 \in \mathbb{R}^{n+1}$. This map is a bijection.

Now it holds

$$\tilde{x}_i = \frac{Rx_i}{R + x_{n+1}},\ i = 1,\dots,n,\ \tilde{r}^2 = \sum_{i=1}^n \tilde{x}_i^2 = \frac{x_{n+1} - R}{x_{n+1} + R}R^2 < R^2\,,$$

$$x_i = \frac{2R^2\tilde{x}_i}{R^2 - \tilde{r}^2},\ i = 1,\dots,n,\ x_{n+1} = \frac{R^2 + \tilde{r}^2}{R^2 - \tilde{r}^2}R\,.$$

Under this map the restriction of the Minkowski metric (1.23) onto the sheet of hyperboloid with $x_{n+1} \geqslant R$ is transformed into the metric

$$g_h = 4R^4\sum_{i=1}^n d\tilde{x}_i^2 \bigg/ \left(R^2 - \tilde{r}^2\right)^2,$$

and the substitution $\tilde{x}_i =: Ry_i$ leads to

$$g_h = 4R^2\sum_{i=1}^n dy_i^2 \Bigg/ \left(1 - \sum_{i=1}^n dy_i^2\right)^2,\ \sum_{i=1}^n y_i^2 < 1. \tag{1.24}$$

This model of the hyperbolic space in the unit ball is known as the *Poincaré* one.

The same metric in the spherical form is

$$g_h = 4R^2\left(dr^2 + r^2\tilde{g}_s\right)/\left(1 - r^2\right)^2,\ r < 1\,, \tag{1.25}$$

where $\tilde{g}_s$ is the same metric as in (1.19).

The Euclidean sphere defined in P_1 by the equation $r = 1$ is called the *absolute*. The arcs of Euclidean circles or segments of lines that are orthogonal (in Euclidean sense) to the absolute are geodesics in the hyperbolic geometry.

For the coordinate $v = 2r/(1 + r^2)$ one gets

$$g_h = R^2\left(\frac{dv^2}{(1 - v^2)^2} + \frac{v^2\tilde{g}_s}{1 - v^2}\right). \tag{1.26}$$

This form of the metric corresponds to the stereographic projection of the same sheet from the point $0 \in \mathbb{R}^{n+1}$ onto the subspace $P_2 = (\mathbf{x} \in \mathbb{R}^{n+1}|\, x_{n+1} = R)$, where vR is the Euclidean distance between a point from P_2 and the fixed point $\mathbf{x}'' = (0, \ldots, 0, R)$. Straight lines in P_2 are again geodesics of metric (1.26) since geodesics of metric (1.23) on the hyperboloid are cut by two dimensional planes in $\mathbb{R}^{n+1}$ passing through $0 \in \mathbb{R}^{n+1}$. This model of the hyperbolic space in the unit ball is known as the *Beltrami-Klein* one.

For the distance ρ from the point $r = 0$ in the metric (1.25) one obtains the formulas:

$$r = \tanh\left(\frac{\rho}{2R}\right),\ v = \tanh\left(\frac{\rho}{R}\right). \tag{1.27}$$

There is known also the *Poincaré model of the hyperbolic space in the upper half space*:

$$\mathbb{R}^n_+ := (\mathbf{x} \in \mathbb{R}^n|\, x_n > 0)\ , \tag{1.28}$$

with the metric:

$$g_h = \frac{R^2}{x_n^2}\sum_{i=1}^{n} dx_i^2\,.$$

In this model the lines $x_i = c_i = \text{const}$, $i = 1, 2, \ldots, n-1$ and the arcs of Euclidean circles that are orthogonal (in Euclidean sense) to the hyperplane, defined in $\mathbb{R}^n$ by the equation $x_n = 0$, are geodesics. The union of this hyperplane and the point at infinity is the absolute.

The group of all linear transformations of the space $\mathbb{R}^{n+1}$, conserving metric (1.23), is called *pseudoorthogonal group* and is denoted by $\mathrm{O}(1, n)$. It consists of four connected components. Two of these components map one sheet of the two-sheet hyperboloid (1.22) into another one. Two other components form the isometry group for $\mathbf{H}^n(\mathbb{R})$. Let $\mathrm{O}_0(1, n)$ be the identity component for $\mathrm{O}(1, n)$. It acts transitively on $\mathbf{H}^n(\mathbb{R})$ with a stationary subgroup isomorphic to $\mathrm{SO}(n)$.

There are three types of actions of one parameter subgroups $\exp(tX)$, $X \in \mathfrak{g}, t \in \mathbb{R}$ of the group $\mathrm{O}_0(1, n)$ in the hyperbolic space $\mathbf{H}^n(\mathbb{R})$ [11]. A one parameter subgroup, isomorphic to $\mathbf{S}^1$, conserves all points of some completely geodesic submanifold, isomorphic to $\mathbf{H}^{n-2}(\mathbb{R})$ for $n \geqslant 4$, and is called *rotation*. In small dimension it conserves all points of some geodesic (for $n = 3$)

or some point (for $n = 2$) and is called rotation about this geodesic (*an axis of a rotation*) for $n = 3$ or about this point (*a center of a rotation*) for $n = 2$. The corresponding element X is called *elliptic*. Euclidean rotations in two-dimensional Euclidean planes passing through the point $y_i = 0$, $i = 1, \dots, n$ are examples of such transformations for the model (1.24). The similar rotations in two dimensional Euclidean planes parallel to the hyperplane, defined in $\mathbb{R}^n$ by the equation $x_n = 0$, are examples of such transformations for the model (1.28).

If a one-parameter subgroup of $\mathrm{O}_0(1, n)$, isomorphic to $\mathbb{R}$, conserves some geodesic then it is called a *transvection* along this geodesic (*an axis of a transvection*). The corresponding element $X \in \mathfrak{so}(1, n)$ is called *hyperbolic*. The dilation $x_i \to e^t x_i$, $t \in \mathbb{R}$ for the model (1.28) is an example of such transformations.

The last type of action of a one-parameter subgroup of $\mathrm{O}_0(1, n)$, is a parabolic action of $\mathbb{R}$. An example of this transformation for the model (1.28) is the Euclidean shift $\mathbf{x} \to \mathbf{x} + t\mathbf{c}$, $t \in \mathbb{R}$, where $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{c} \in \mathbb{R}^n$, $c_n = 0$. For $\mathbf{H}^2(\mathbb{R})$ it shifts points along the system of *horocycles* that are lines orthogonal at each point to all geodesics having a common point on the absolute. The corresponding element $X \in \mathfrak{so}(1, n)$ is called *parabolic*.

Due to the transitivity of an isometry group an every elliptic, hyperbolic or parabolic transformation is conjugated with transformations from these examples.

One can formally transform many formulas valid for the spherical metric (1.16) into formulas valid for the hyperbolic metric (1.24) by the substitution

$$y_i \to \mp \mathbf{i} y_i,\ r \to \mp \mathbf{i} r,\ v \to \mp \mathbf{i} v,\ R \to \pm \mathbf{i} R\ , \tag{1.29}$$

due to well-known formulas

$$\cos(\mathbf{i}x) = \cosh x, \sin(\mathbf{i}x) = \mathbf{i} \sin x,\ \arctan(\mathbf{i}x) = \mathbf{i}\,\mathrm{arctanh}\, x. \tag{1.30}$$

This transition corresponds to the transition described in Proposition 1.5. Note that the factor $\pm\mathbf{i}$ in Proposition 1.5 correspond to the factor $\mp\mathbf{i}$ in (1.29) since elements of Lie algebras correspond to vector fields, which are differential operators of the first order.

The similar correspondence is valid for other two-point homogeneous spaces.

1.4 The Model of the Projective Cayley Plane

Our description of the Cayley algebra $\mathbb{C}a$ and the octonionic projective plane $\mathbf{P}^2(\mathbb{C}a)$ in this section is based upon [2, 10] and [142].

1.4.1 The Algebra $\mathbb{C}a$

According to *Frobenius theorem* there are only four finite-dimensional division algebras over $\mathbb{R}$: $\mathbb{R}$ itself and algebras $\mathbb{C}, \mathbb{H}, \mathbb{C}a$. The latter is an eight dimensional normed division algebra of octonions. It is noncommutative and nonassociative, but alternative, i.e., for any two elements $\xi, \eta \in \mathbb{C}a$ it holds $(\xi\eta)\eta = \xi(\eta\eta)$ and $\xi(\xi\eta) = (\xi\xi)\eta$. The group of all automorphisms of $\mathbb{C}a$ is the *exceptional simple compact real* 14*-dimensional Lie group* G_2. The standard base of $\mathbb{C}a$ over $\mathbb{R}$ is $(e_i)_{i=0}^{7}$, where $e_0 = 1 \in \mathbb{R}$ and $e_i^2 = -1$, $e_ie_j = -e_je_i$, $i, j = 1, \ldots, 7$, $i \neq j$. The elements $(e_i)_{i=1}^{7}$ are multiplied according to the following scheme:

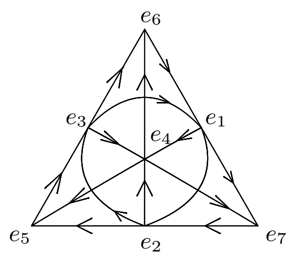

Fig. 1.1.

Here, $e_ie_j = e_k$ if these elements lie on one line or on the circle and are ordered by arrows as e_i, e_j, e_k. The *octonionic conjugation* $\iota : \mathbb{C}a \mapsto \mathbb{C}a$ acts as $\iota(e_0) \equiv \overline{e}_0 = e_0$, $\iota(e_i) \equiv \overline{e}_i = -e_i, i = 1, \ldots, 7$ and is extended by linearity over whole $\mathbb{C}a$. Let $\mathrm{Re}\,\xi := \frac{1}{2}(\xi + \bar{\xi})$, $\mathrm{Im}\,\xi := \frac{1}{2}(\xi - \bar{\xi})$ be the real and the imaginary parts of $\xi \in \mathbb{C}a$. Define the scalar product in $\mathbb{C}a$ by the formula: $\langle \eta, \xi \rangle = \dfrac{1}{2}(\bar{\eta}\xi + \bar{\xi}\eta) = \mathrm{Re}(\bar{\xi}\eta) = \mathrm{Re}(\bar{\eta}\xi) \in \mathbb{R}$ and the norm by the formula $\|\eta\| = \langle \eta, \eta \rangle^{1/2}$. In the algebra $\mathbb{C}a$ every two elements generate an associative subalgebra and the following *central Moufang identity* is valid:

$$u \cdot xy \cdot u = ux \cdot yu,\; u, x, y \in \mathbb{C}a\ . \tag{1.31}$$

Here we use the notations $u \cdot xy := u(xy)$, $xy \cdot u := (xy)u$.

There are the following description of left and right spinor representations (both 8-dimensional) of the group Spin(8) in $\mathbb{C}a$ [134, 142], which will be used later. Define linear operators in $\mathbb{C}a$:

$$L_\alpha : \xi \mapsto \frac{1}{2}e_\alpha\xi,\; \alpha = 1, \ldots, 7,\; \xi \in \mathbb{C}a\ ,$$
$$L_{\alpha,\beta} : \xi \mapsto \frac{1}{2}e_\alpha(e_\beta\xi),\; 1 \leqslant \alpha < \beta \leqslant 7,\; \xi \in \mathbb{C}a\ .$$

These operators are generators of the left spinor representation of the group Spin(8), i.e., they are the images of some base of the Lie algebra $\mathfrak{spin}(8)$ under this representation. Similarly, operators

$$R_\alpha : \xi \mapsto \frac{1}{2}\xi e_\alpha,\; \alpha = 1, \ldots, 7,\; \xi \in \mathbb{C}a\ ,$$
$$R_{\alpha,\beta} : \xi \mapsto \frac{1}{2}(\xi e_\beta)e_\alpha,\; 1 \leqslant \alpha < \beta \leqslant 7,\; \xi \in \mathbb{C}a$$

are generators of the right spinor representation of the group Spin(8). All these operators are skew symmetric w.r.t. the scalar product in $\mathbb{C}a$.

Formulae above define operators $L_{\alpha,\beta}, R_{\alpha,\beta}$ also for $1 \leqslant \beta < \alpha \leqslant 7$. If $\mathbb{C}a'$ is the space of pure imaginary octonions, $u \in \mathbb{C}a'$, $\xi \in \mathbb{C}a$, then due to the alternativity of $\mathbb{C}a$:

$$\xi u \cdot u = \xi u^2 = -\xi|u|^2 = -|u|^2\xi = u \cdot u\xi \ .$$

For $u = e_\alpha + e_\beta$, $1 \leqslant \alpha < \beta \leqslant 7$ it holds

$$\begin{aligned} -2\xi &= -\xi|e_\alpha + e_\beta|^2 = \xi(e_\alpha + e_\beta) \cdot (e_\alpha + e_\beta) \\ &= \xi e_\alpha \cdot e_\alpha + \xi e_\alpha \cdot e_\beta + \xi e_\beta \cdot e_\alpha + \xi e_\beta \cdot e_\beta = -\xi + \xi e_\alpha \cdot e_\beta + \xi e_\beta \cdot e_\alpha - \xi \end{aligned}$$

and $\xi e_\alpha \cdot e_\beta + \xi e_\beta \cdot e_\alpha = 0$. Similarly, $e_\alpha \cdot e_\beta \xi + e_\beta \cdot e_\alpha \xi = 0$. For $0 \leqslant i, j \leqslant 7$, $i \neq j$ we can write more general formulae:

$$e_i \cdot e_j\xi = -\bar{e}_j \cdot \bar{e}_i\xi,\ \xi e_i \cdot e_j = -\xi\bar{e}_j \cdot \bar{e}_i,\ \xi \in \mathbb{C}a \ , \tag{1.32}$$

useful in the followings.

In particular, we have $L_{\alpha,\beta} = -L_{\beta,\alpha}$, $R_{\alpha,\beta} = -R_{\beta,\alpha}$, $1 \leqslant \alpha, \beta \leqslant 7, \alpha \neq \beta$.

The group Spin(8) is the double covering of the group SO(8) and the tautological representation of the latter induced the *vector representation* of Spin(8), evidently also 8-dimensional. For the element $q \in \mathrm{Spin}(8)$ denote by q^L, q^R and q^V its images under left spinor, right spinor and vector representation respectively. The following proposition is a version of the *triality principle* for the group Spin(8)[5].

Proposition 1.7 ([142]). *For any element $q \in \mathrm{Spin}(8)$ it holds*

$$q^V(\xi\eta) = q^L(\xi) \cdot q^R(\eta),\ \xi, \eta \in \mathbb{C}a \ . \tag{1.33}$$

Conversely, if A, B, C are orthogonal operators $\mathbb{C}a \mapsto \mathbb{C}a$ such that

$$A(\xi\eta) = B(\xi) \cdot C(\eta) \ ,$$

for any $\xi, \eta \in \mathbb{C}a$, then there exist a unique $q \in \mathrm{Spin}(8)$ such that $A = q^V, B = q^L, C = q^R$.

From (1.33) we get its infinitesimal analogs:

$$\begin{aligned} V_i(\xi\eta) &= L_i(\xi) \cdot \eta + \xi \cdot R_i(\eta),\ i = 1, \ldots, 7 \ , \\ V_{i,j}(\xi\eta) &= L_{i,j}(\xi) \cdot \eta + \xi \cdot R_{i,j}(\eta),\ 1 \leqslant i < j \leqslant 7, \xi, \eta \in \mathbb{C}a \ , \end{aligned} \tag{1.34}$$

where V_i and $V_{i,j}$ are generators of the vector representation of the group Spin(8).

1.4.2 The Jordan Algebra $\mathfrak{h}_3(\mathbb{C}a)$

The *Hermitian conjugation* $A \mapsto A^*$ for square matrix with octonion entries is defined as the composition of the octonionic conjugation and the transposition of A, similar to complex and quaternion cases. A matrix A is called

[5] Other versions of this principle can be found in [10].

Hermitian iff $A^* = A$. The *simple exceptional Jordan algebra* $\mathfrak{h}_3(\mathbb{C}a)$ consists of all Hermitian 3×3 matrices with octonion entries. It is endowed with the Jordan commutative multiplication:

$$X \circ Y = \frac{1}{2}(XY + YX),\ X, Y \in \mathfrak{h}_3(\mathbb{C}a)\ .$$

This multiplication satisfies the identity $(X^2 \circ Y) \circ X = X^2 \circ (Y \circ X)$, which is the condition for an algebra with commutative (but not necessarily associative) multiplication to be Jordan. The Jordan algebra $\mathfrak{h}_3(\mathbb{C}a)$ is 27-dimensional over $\mathbb{R}$. Every its element can be represented in the form:

$$X = a_1E_1 + a_2E_2 + a_3E_3 + X_1(\xi_1) + X_2(\xi_2) + X_3(\xi_3)\ , \tag{1.35}$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$X_1(\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & \bar{\xi} & 0 \end{pmatrix},\ X_2(\xi) = \begin{pmatrix} 0 & 0 & \bar{\xi} \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix},\ X_3(\xi) = \begin{pmatrix} 0 & \xi & 0 \\ \bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$a_i \in \mathbb{R}, \xi_i \in \mathbb{C}a,\ i = 1, 2, 3$. It is easy to show that

$$\begin{aligned} E_i \circ E_j &= \begin{cases} E_i, \text{ if } i = j, \\ 0, \text{ if } i \neq j, \end{cases} \\ E_i \circ X_j(\xi) &= \begin{cases} 0, \text{ if } i = j, \\ \frac{1}{2}X_j(\xi), \text{ if } i \neq j, \end{cases} \\ X_i(\xi) \circ X_j(\eta) &= \begin{cases} \langle \xi, \eta \rangle (E - E_i), \text{ if } i = j, \\ \frac{1}{2}X_{i+j}(\overline{\xi\eta}), \text{ if } j \equiv i + 1 \mod 3, \end{cases} \end{aligned} \tag{1.36}$$

where $E = E_1 + E_2 + E_3$ is the unit matrix. In the last formula all indices are considered modulo 3.

The group of all automorphisms of the Jordan algebra $\mathfrak{h}_3(\mathbb{C}a)$ is the simple compact 52-dimensional *exceptional real Lie group* F_4. This group conserves the following bilinear and trilinear forms: $\mathcal{A}(X, Y) = \mathrm{Tr}(X \circ Y)$, $\mathcal{B}(X, Y, Z) = \mathcal{A}(X \circ Y, Z)$. Conversely, every linear operator $\mathfrak{h}_3(\mathbb{C}a) \mapsto \mathfrak{h}_3(\mathbb{C}a)$, conserving these two forms, belongs to F_4.

Define the norm of the element (1.35) as $\|X\|^2 = \mathcal{A}(X, X) = \sum_{i=1}^{3}(a_i^2 + 2|\xi_i|^2)$. The last equality is the consequence of (1.36).

Theorem 1.4 (Freudenthal, [142]). *For any element $X \in \mathfrak{h}_3(\mathbb{C}a)$ there exists an automorphism $\Phi \in F_4$ such that*

$$\Phi X = \lambda_1E_1 + \lambda_2E_2 + \lambda_3E_3\ , \tag{1.37}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$. Expansion (1.37) *is uniquely determined by X. Two elements from $\mathfrak{h}_3(\mathbb{C}a)$ lie on one F_4-orbit, iff their diagonal forms* (1.37) *are the same.*

1.4.3 The Octonionic Projective Plane $\mathbf{P}^2(\mathbb{C}a)$

Elements $X \in \mathfrak{h}_3(\mathbb{C}a)$ satisfying conditions

$$X^2 = X \ , \ \operatorname{Tr} X = 1 \tag{1.38}$$

form the *octonionic projective plane* $\mathbf{P}^2(\mathbb{C}a)$, which is a 16-dimensional real manifold. Automorphisms of $\mathfrak{h}_3(\mathbb{C}a)$ conserves (1.38) and the group F_4 acts on $\mathbf{P}^2(\mathbb{C}a)$. From the Freudenthal theorem and (1.38) it follows that every element of $\mathbf{P}^2(\mathbb{C}a)$ can be transformed by an appropriate element from F_4 into the element E_1. Thus $\mathbf{P}^2(\mathbb{C}a)$ is a homogeneous space of the group F_4 and calculations in [142] (lecture 16) show, that the stationary subgroup of every point $X \in \mathbf{P}^2(\mathbb{C}a)$ is isomorphic to the group Spin(9).

Suppose that an element

$$X = (1 + a_1)E_1 + a_2E_2 + a_3E_3 + X_1(\xi_1) + X_2(\xi_2) + X_3(\xi_3) \in \mathbf{P}^2(\mathbb{C}a)$$

belongs to a neighborhood of E_1 , i.e., real numbers $a_i, |\xi_i|, i = 1, 2, 3$ are sufficiently small. Then due to (1.36) we have

$$X \circ X = (1+2a_1)E_1 + X_2(\xi_2) + X_3(\xi_3) + O(s) \ , \quad \text{where} \quad s := \left(\sum_{i=1}^{3}(a_i^2 + |\xi_i|^2)\right)$$

and the equality $X \circ X = X$ implies $a_i = O(s)$, $i = 1, 2, 3$, $\xi_1 = O(s)$. This means that one can identify the tangent space $T_{E_1}\mathbf{P}^2(\mathbb{C}a)$ with the set $(X_2(\xi_2) + X_3(\xi_3) \,|\, \xi_2, \xi_3 \in \mathbb{C}a)$.

Let $K \subset F_4$ be the stationary subgroup corresponding to the point E_1 and acting by automorphisms in the space $T_{E_1}\mathbf{P}^2(\mathbb{C}a) \simeq (X_2(\xi_2)+X_3(\xi_3) \,|\, \xi_2, \xi_3 \in \mathbb{C}a)$. Let K_0 be the stationary subgroup of K, corresponding to the element $X_3(1) \in T_{E_1}\mathbf{P}^2(\mathbb{C}a)$.

Let us find the K_0-action in the space $T_{E_1}\mathbf{P}^2(\mathbb{C}a)$. For any element $X \in \mathfrak{h}_3(\mathbb{C}a)$ let $\operatorname{ann} X := (Y \in \mathfrak{h}_3(\mathbb{C}a) \,|\, Y \circ X = 0)$. Since an element $\Phi \in K_0$ is an automorphism of the algebra $\mathfrak{h}_3(\mathbb{C}a)$, it conserves the space $\operatorname{ann} X_3(1)$. It follows from (1.36) that

$$\operatorname{ann} X_3(1) = (a(E_1 - E_2) + bE_3 + X_3(\xi) \,|\, a, b \in \mathbb{R},\ \xi \in \mathbb{C}a')\,.$$

Let $\Phi(E_1 - E_2) = a(E_1 - E_2) + bE_3 + X_3(\xi)$, then one has

$$\begin{aligned} 1 = \mathcal{A}\,(E_1 - E_2, E_1) &= \mathcal{A}\,(\Phi(E_1 - E_2), \Phi(E_1)) \\ &= \mathcal{A}\,(a(E_1 - E_2) + bE_3 + X_3(\xi), E_1) = a\ . \end{aligned}$$

This implies $\Phi(E_1 - E_2) = E_1 - E_2 + bE_3 + X_3(\xi)$ and the equality $\|E_1 - E_2\| = \|\Phi(E_1 - E_2)\|$ gives $b = 0, \xi = 0$. It means that $\Phi(E_2) = E_2$ and therefore $\Phi(E_3) = \Phi(E - E_1 - E_2) = E - E_1 - E_2 = E_3$. Thus, the group K_0 conserves elements E_1, E_2, E_3.

Let K' be the maximal subgroup of F_4 conserving elements E_1, E_2, E_3. We see that $K_0 \subset K' \subset K$. Since $\operatorname{ann} E_1 = \{a_2E_2 + a_3E_3 + X_1(\xi),\ a_1, a_2 \in \mathbb{R},\ \xi \in$

$\mathbb{C}a\}$, the group K' maps $X_1(\xi) \mapsto X_1(\tilde{\xi})$ and similarly $X_i(\xi_i) \mapsto X_i(\tilde{\xi}_i)$, $i = 1, 2, 3$.

Let $\Phi_i : \mathbb{C}a \mapsto \mathbb{C}a$, $i = 1, 2, 3$ be orthogonal operators such that $\Phi X_i(\xi_i) = X_i(\Phi_i(\xi_i))$ for $\Phi \in K'$. The last formula in (1.36) implies

$$X_3\left(\Phi_3(\overline{\xi\eta})\right) = \Phi\left(X_3(\overline{\xi\eta})\right) = 2\Phi\left(X_1(\xi) \circ X_2(\eta)\right) = 2\Phi\left(X_1(\xi)\right) \circ \Phi\left(X_2(\eta)\right)$$
$$= 2X_1\left(\Phi_1(\xi)\right) \circ X_2\left(\Phi_2(\eta)\right) = X_3\left(\overline{\Phi_1(\xi)\Phi_2(\eta)}\right).$$

This gives

$$\Phi_1(\xi)\Phi_2(\eta) = \overline{\Phi_3\left(\overline{\xi\eta}\right)}, \tag{1.39}$$

for $\Phi \in K'$, $\xi, \eta \in \mathbb{C}a$.

Denote by $\mathbb{C}a_i$, $i = 1, 2, 3$ the domains of the operators Φ_i, $i = 1, 2, 3$. Then one can identify the space $T_{E_1}\mathbf{P}^2(\mathbb{C}a)$ with $\mathbb{C}a_2^* \oplus \mathbb{C}a_3^*$, where the superscript $*$ denotes a dual space.[6] Formula (1.39) and Proposition 1.7 imply.

Proposition 1.8. *Operators Φ_1 and Φ_2 are respectively left and right spinor representations of the group* Spin(8) $\simeq K'$ *and the composition $\iota \circ \Phi_3 \circ \iota$ is the vector representation of* Spin(8).

Since the group Spin(8) is the universal (double) covering of the group SO(8), the Lie algebras $\mathfrak{spin}(8)$ and $\mathfrak{so}(8)$ are isomorphic.

Now consider representations of the Lie algebra $\mathfrak{k}'$ of the group K' in $\mathbb{C}a_i$, $i = 1, 2, 3$. All these representations are faithful. For $A \in \mathfrak{k}'$ denote by $A^{(i)}$ the corresponding skew symmetric operator in $\mathbb{C}a_i$, $i = 1, 2, 3$. From (1.39) one gets the following *infinitesimal analogue of the triality principle*:

$$A^{(1)}(\xi) \cdot \eta + \xi \cdot A^{(2)}(\eta) = \overline{A^{(3)}(\overline{\xi\eta})}. \tag{1.40}$$

From (1.34) we obtain that if $A^{(1)} = L_i$ (respectively $A^{(1)} = L_{i,j}$), then $A^{(2)} = R_i$, $A^{(3)} = \iota \circ V_i \circ \iota$ (respectively $A^{(2)} = R_{i,j}$, $A^{(3)} = \iota \circ V_{i,j} \circ \iota$).

Let us identify the algebra $\mathfrak{k}'$ with its vector representation in $\mathbb{C}a_3$, in particular we put $A \equiv A^{(3)}$ for $A \in \mathfrak{k}'$. By $\varkappa$ denote the inclusion of $\mathfrak{k}'$ into the Lie algebra $\mathfrak{f}_4$ corresponding to the group F_4.

By the definition, the Lie algebra $\mathfrak{k}_0$ of the group $K_0 \subset K'$ consists of the skew symmetric operators in $\mathbb{C}a_3$, transforming $1 \in \mathbb{C}a_3$ into 0. Therefore, from above one gets the following proposition.

Proposition 1.9. *The group K_0 is isomorphic to* Spin(7) , *acting in $\mathbb{C}a_1$ by the left spinor representation, in $\mathbb{C}a_2$ by the right spinor representation (equivalent for* Spin(7) *to the left one, see (3.40) below), and in $\mathbb{C}a_3'$ by the vector representation. These representations are restrictions of corresponding representations of the group $K' \simeq$* Spin(8).

Let $\mathfrak{m}$ be a space of 3×3 skew-Hermitian matrices with octonion entries and the zero trace. Let

[6] Here we consider octonions $\xi \in \mathbb{C}a_2$ and $\eta \in \mathbb{C}a_3$ as coordinates of elements $X_2(\xi)$ and $X_3(\eta)$ respectively.

$$Y_1(\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\bar{\xi} & 0 \end{pmatrix}, \; Y_2(\xi) = \begin{pmatrix} 0 & 0 & -\bar{\xi} \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix}, \; Y_3(\xi) = \begin{pmatrix} 0 & \xi & 0 \\ -\bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \xi \in \mathbb{C}a$$

be elements from $\mathfrak{m}$. Assume also that the linear subspace $\mathfrak{m}_0 \subset \mathfrak{m}$ consists of elements of the form

$$\sum_{i=1}^{3} Y_i(\xi_i),\ \xi_i \in \mathbb{C}a\ .$$

From [142] (lecture 16) we can extract the following proposition:

Proposition 1.10. *For any $Y \in \mathfrak{m}$ the linear operator $\operatorname{ad} Y : \mathfrak{h}_3(\mathbb{C}a) \mapsto \mathfrak{h}_3(\mathbb{C}a)$, acting according to the formula $\operatorname{ad} Y(X) = YX - XY$, $X \in \mathfrak{h}_3(\mathbb{C}a)$ is a differentiation of the algebra $\mathfrak{h}_3(\mathbb{C}a)$. Thus, the space $\mathfrak{m}$ is contained in the Lie algebra $\mathfrak{f}_4$. There is the expansion of $\mathfrak{f}_4$ into the direct sum of linear spaces*

$$\mathfrak{f}_4 \simeq \mathfrak{k}' \oplus \mathfrak{m}_0$$

with the following commutator relations

$$[\varkappa A, \operatorname{ad} Y_i(\xi)] = \operatorname{ad} Y_i(A^{(i)}\xi),\ i = 1,2,3\ , \tag{1.41}$$

$$[\operatorname{ad} Y_i(\xi), \operatorname{ad} Y_j(\eta)] = \begin{cases} \varkappa C_{i,\xi,\eta},\ \text{if}\ j = i\ , \\ \operatorname{ad} Y_{i+2}\left(-\overline{\xi\eta}\right),\ \text{if}\ j = i+1\ , \end{cases} \tag{1.42}$$

where $A \equiv A^{(3)} \in \mathfrak{k}'$, $\xi, \eta \in \mathbb{C}a$, operators $A^{(i)}$ are from (1.40)*, the indices in* (1.42) *are considered modulo* 3 *and skew-Hermitian operators $C_{i,\xi,\eta} : \mathbb{C}a_3 \mapsto \mathbb{C}a_3$, $i = 1,2,3$ are given by the following formulas:*

$$\begin{aligned} C_{1,\xi,\eta} &: \zeta \mapsto \zeta\xi \cdot \bar{\eta} - \zeta\eta \cdot \bar{\xi}\ , \\ C_{2,\xi,\eta} &: \zeta \mapsto \bar{\eta} \cdot \xi\zeta - \bar{\xi} \cdot \eta\zeta,\ \zeta \in \mathbb{C}a \\ C_{3,\xi,\eta} &: \zeta \mapsto 4\langle \xi, \zeta\rangle\eta - 4\langle \eta, \zeta\rangle\xi\ . \end{aligned} \tag{1.43}$$

The action of operators $\varkappa C_{i,\xi,\eta}$ in the spaces $\mathbb{C}a_1$ and $\mathbb{C}a_2$ is obtained from (1.43) by the cyclic permutation of indices:

$$\begin{aligned} \varkappa C_{1,\xi,\eta}|_{\mathbb{C}a_1} &: \zeta \mapsto 4\langle \xi, \zeta\rangle\eta - 4\langle \eta, \zeta\rangle\xi\ , \\ \varkappa C_{2,\xi,\eta}|_{\mathbb{C}a_1} &: \zeta \mapsto \zeta\xi \cdot \bar{\eta} - \zeta\eta \cdot \bar{\xi}\ , \\ \varkappa C_{3,\xi,\eta}|_{\mathbb{C}a_1} &: \zeta \mapsto \bar{\eta} \cdot \xi\zeta - \bar{\xi} \cdot \eta\zeta\ , \\ \varkappa C_{1,\xi,\eta}|_{\mathbb{C}a_2} &: \zeta \mapsto \bar{\eta} \cdot \xi\zeta - \bar{\xi} \cdot \eta\zeta\ , \\ \varkappa C_{2,\xi,\eta}|_{\mathbb{C}a_2} &: \zeta \mapsto 4\langle \xi, \zeta\rangle\eta - 4\langle \eta, \zeta\rangle\xi\ , \\ \varkappa C_{3,\xi,\eta}|_{\mathbb{C}a_2} &: \zeta \mapsto \zeta\xi \cdot \bar{\eta} - \zeta\eta \cdot \bar{\xi}\ . \end{aligned} \tag{1.44}$$

Note that in [142] (lecture 16) analogs of formulae (1.39), (1.41) and the last formula (1.36) contain errors.

The volume function on the octonion projective plane $\mathbf{P}^2(\mathbb{C}a)$ obeys the following equalities [55, 66]:

$$\begin{aligned}
\mathrm{vol}\left(B_{\mathbf{P}^2(\mathbb{C}a)}(\rho)\right) &= \frac{6\pi^8(2R)^{16}}{11!}\sin^{16}\left(\frac{\rho}{2R}\right)\left(1+8\cos^2\left(\frac{\rho}{2R}\right)+36\cos^4\left(\frac{\rho}{2R}\right)\right.\\
&\quad\left.+120\cos^6\left(\frac{\rho}{2R}\right)\right) ,\\
\mathrm{vol}\left(\mathbf{S}_{\mathbf{P}^2(\mathbb{C}a)}(\rho)\right) &= \frac{d}{d\rho}\,\mathrm{vol}\left(B_{\mathbf{P}^2(\mathbb{C}a)}(\rho)\right) = \frac{2^9\pi^8R^{15}}{7!}\sin^8\left(\frac{\rho}{2R}\right)\sin^7\left(\frac{\rho}{R}\right) ,\\
\mathrm{vol}\left(\mathbf{P}^2(\mathbb{C}a)\right) &= \mathrm{vol}\left(B_{\mathbf{P}^2(\mathbb{C}a)}(\pi R)\right) = \frac{6\pi^8(2R)^{16}}{11!} .
\end{aligned}$$

The description of the octonion projective plane $\mathbf{P}^2(\mathbb{C}a)$ through the Jordan algebra $\mathfrak{h}_3(\mathbb{C}a)$ is not an exceptional case in the theory of two-point homogeneous spaces. Other compact two-point homogeneous spaces also could be described through corresponding Jordan algebras [10]. Therefore, this description seems to be more fundamental than the models in Euclidean spaces in the Sect. 1.3, but also more cumbersome.

2

Differential Operators on Smooth Manifolds

The property of a differential operator on a smooth manifold M to be invariant with respect to an action of some group G (especially a Lie group) on M plays a great role in mathematical physics since it helps select physically significant operators. The algebra $\mathrm{Diff}_G(M)$ of all G-invariant differential operators with complex or real coefficients on M gives the material for constructing G-invariant physical theories on M. Properties of such theory are in close connection with properties of the algebra $\mathrm{Diff}_G(M)$.

A homogeneous smooth manifold M of the Lie group G is called *commutative space*, if the algebra $\mathrm{Diff}_G(M)$ is commutative. The well-known example of a commutative space is the symmetric space of the rank l. The commutative algebra $\mathrm{Diff}_G(M)$ for this space is generated by l independent commutative generators [64]. Particularly, for symmetric spaces of the rank one and therefore for two-point homogeneous Riemannian spaces the algebra $\mathrm{Diff}_G(M)$ is generated by the Laplace–Beltrami operator. Also, the class of commutative spaces contains weakly symmetric spaces [199].

There are known only some sporadic examples of noncommutative algebras $\mathrm{Diff}_G(M)$ (see, for example, Chap. 2 from [66]). One of these example is the noncommutative algebra $\mathrm{Diff}_{\mathrm{O}_0(1,n)}(M_1)$ for $M_1 = \mathrm{O}_0(1,n)/\,\mathrm{SO}(n-1)$ described in [146], where the space M_1 was interpreted as the total space for the unit sphere bundle over the hyperbolic space $\mathbf{H}^n(\mathbb{R})$. Other examples of noncommutative algebras $\mathrm{Diff}_G(M)$ were considered in [168] and are described below in Chap. 3.

2.1 Invariant Differential Operators on Lie Groups and Homogeneous Manifolds

2.1.1 Basic Notations

This section contains basic facts from the theory of invariant differential operators on homogeneous manifolds [62, 64, 66, 67]. The main results are the following: an algebra of invariant differential operators on G-homogeneous manifolds is finitely generated over the basic field; it is described in terms

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of the universal enveloping algebra $U(\mathfrak{g})$; there is a one-to-one linear correspondence between invariant differential operators and their symbols, defined below as elements of the symmetric algebra $S(\mathfrak{g})$.

For our purposes it is enough to suppose that the basic field is $\mathbb{R}$, however most results in this section are valid also for the basic field $\mathbb{C}$.

Let G be a connected Lie group, $\dim G = N$ and M be a smooth G-homogeneous manifold, $\dim M = m$. Let also $K_{x_0} \subset G$ be the stationary subgroup of a point $x_0 \in M$ and $\mathfrak{k}_{x_0} \subset \mathfrak{g} \equiv T_eG$ be the corresponding Lie subalgebra, where $e \in G$ is the unit element. Choose a subspace $\mathfrak{p}_{x_0} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p}_{x_0} \oplus \mathfrak{k}_{x_0}$ (a direct sum of linear spaces). From now the point x_0 is fixed in this section and we omit the index x_0 in notations for $K_{x_0}, \mathfrak{k}_{x_0}, \mathfrak{p}_{x_0}$ and so on.

Identify the space M with the factor space G/K of left cosets. Let $\pi_1 : G \to G/K$ be the natural projection. Denote by

$$L_q : q_1 \to qq_1, \quad R_q : q_1 \to q_1 q, \quad q, q_1 \in G$$

the left and the right shifts on the group G and by

$$\tau_q : x \to qx, \quad q \in G, \quad x \in M$$

the action of an element $q \in G$ on M. Obviously, $\pi_1 \circ L_q = \tau_q \circ \pi_1, \quad q \in G$ and $\pi_1 \circ R_q = \pi_1$, $q \in K$. Let the left and the right shifts act on the space $C^\infty(G)$ as:

$$\hat{L}_q(f)(q_1) = f(q^{-1}q_1), \quad \hat{R}_q(f)(q_1) = f(q_1 q^{-1}), \quad f \in C^\infty(G),\, q, q_1 \in G\,.$$

Similarly, the left shift acts on the space $C^\infty(M)$ as:

$$\hat{\tau}_q(f)(x) = f(q^{-1}x), \quad f \in C^\infty(M),\, q, q_1 \in G\,.$$

Then it holds $\hat{L}_{q_1 q_2} = \hat{L}_{q_1} \circ \hat{L}_{q_2}, \quad \hat{R}_{q_1 q_2} = \hat{R}_{q_2} \circ \hat{R}_{q_1}, \quad \hat{\tau}_{q_1 q_2} = \hat{\tau}_{q_1} \circ \hat{\tau}_{q_2}$, $\hat{L}_{q_1} \circ \hat{R}_{q_2} = \hat{R}_{q_2} \circ \hat{L}_{q_1}$, $q_1, q_2 \in G$.

Let $\mathrm{Diff}(G)$ and $\mathrm{Diff}(M)$ be algebras of differential operators with smooth real coefficients on G and M respectively. Define the action of shifts on operators as:

$$\tilde{L}_q(\Box) = \hat{L}_q \circ \Box \circ \hat{L}_{q^{-1}}, \quad \tilde{R}_q(\Box) = \hat{R}_q \circ \Box \circ \hat{R}_{q^{-1}}, \quad \Box \in \mathrm{Diff}(G)\,,$$
$$\tilde{\tau}_q(\Box) = \hat{\tau}_q \circ \Box \circ \hat{\tau}_{q^{-1}}, \quad \Box \in \mathrm{Diff}(M)\,.$$

Define also the following subalgebras:

$$\begin{aligned}
\mathrm{LDiff}(G) &:= \left\{\Box \in \mathrm{Diff}(G) |\, \tilde{L}_q(\Box) = \Box,\, \forall q \in G\right\}\,,\\
\mathrm{Diff}_G(M) &:= \{\Box \in \mathrm{Diff}(M) |\, \tilde{\tau}_q(\Box) = \Box,\, \forall q \in G\}\,,\\
\mathrm{RDiff}(G) &:= \left\{\Box \in \mathrm{Diff}(G) |\, \tilde{R}_q(\Box) = \Box,\, \forall q \in G\right\}\,,\\
\mathrm{LRDiff}(G) &:= \left\{\Box \in \mathrm{LDiff}(G) |\, \tilde{R}_q(\Box) = \Box,\, \forall q \in G\right\}\,,\\
\mathrm{LDiff}_K(G) &:= \left\{\Box \in \mathrm{LDiff}(G) |\, \tilde{R}_q(\Box) = \Box,\, \forall q \in K\right\}\,.
\end{aligned}$$

An algebra A is called *filtered* if there is a sequence $\{F_i\}_{i=1}^{\infty}$ of its subspaces such that

$$F_1 \subset F_2 \subset \cdots \subset A = \bigcup_{i=1}^{\infty} F_i, \ F_i F_j \subset F_{i+j} \ .$$

An algebra A is called *graded* if there is a sequence $\{A_i\}_{i=1}^{\infty}$ of its subspaces such that $A = \bigoplus_{i=1}^{\infty} A_i$ and $A_i A_j \subset A_{i+j}$. For a filtered algebra A the corresponding graded algebra $\operatorname{gr} A$ is $\bigoplus_{i=1}^{\infty} F_i/F_{i-1}$. If $a \in F_i/F_{i-1}, b \in F_j/F_{j-1}$ then the product ab evidently belongs to F_{i+j}/F_{i+j-1}.

A *degree* of an element a of a filtered algebra A equals i iff $a \in F_i \setminus F_{i-1}$. Algebras $\mathrm{Diff}(M)$ and $\mathrm{Diff}(G)$ have the standard filtrations for which spaces $\mathrm{Diff}(M)_i$ and $\mathrm{Diff}(G)_i$ consist of differential operators of degree $\leqslant i$. For any algebra $\mathcal{A}$ denote by $\mathrm{Z}\,\mathcal{A}$ the center of $\mathcal{A}$.

Let $e_1, \ldots, e_N$ be a base in $\mathfrak{g}$ such that $e_1, \ldots, e_m$ is a base in $\mathfrak{p}$ and $e_{m+1}, \ldots, e_N$ is a base in $\mathfrak{k}$. There are corresponding moving frames on the group G consisting respectively of the following left and right invariant vector fields:

$$X_i^l(q) = (dL_q)e_i, \quad X_i^r(q) = (dR_q)e_i, \, i = 1, \ldots, N, \, q \in G \ .$$

There are also the moving coframes $X_l^i(q), X_r^i(q)$. For an element $Y \in \mathfrak{g}$ we shall denote by Y^l and Y^r the corresponding left- and right-invariant vector fields on G and by $\widetilde{Y}$ the corresponding vector field on M : $\widetilde{Y} = \frac{d}{dt} \exp(tY)x\big|_{t=0}$, $x \in M$. Evidently, it holds $d\pi_1(Y^r) = \widetilde{Y}$. Note that left invariant vector fields correspond to right G-shifts on G and conversely right invariant vector fields correspond to left G-shifts on G.

Define the *commutator of vector fields*[1] $X', Y' \in \mathcal{X}(M)$ through their action on functions $f \in C^{\infty}(M)$:

$$[X', Y']f = X'(Y'f) - Y'(X'f) \ .$$

It is well known that the correspondence $X \to \widetilde{X}$, $X \in \mathfrak{g}$ changes signs of commutators (see, for example, [17], 7.21):

$$\widetilde{[X, Y]} = -[\widetilde{X}, \widetilde{Y}] \ . \tag{2.1}$$

In particular

$$[X, Y]^r = -[X^r, Y^r] \ . \tag{2.2}$$

On the other hand, a right action of a Lie group conserves signs of commutators. In particular for the right action of G on itself one has:

$$[X, Y]^l = [X^l, Y^l] \ . \tag{2.3}$$

[1] For any smooth manifold M, not necessarily endowed with a G-action.

2.1.2 Invariant Differential Operators on a Lie Group

Let $U_1(\mathfrak{g}) \subset U_2(\mathfrak{g}) \subset \cdots \subset U(\mathfrak{g})$ be the standard filtration of the universal enveloping algebra $U(\mathfrak{g})$ for $\mathfrak{g}$, where $U_k(\mathfrak{g})$ consists of elements of the form $P_i(e_1, \ldots, e_N)$, $1 \leqslant i \leqslant k$; P_i is a polynomial of degree i without a constant term[2]. Since $[U_i(\mathfrak{g}), U_j(\mathfrak{g})] \subset U_{i+j-1}(\mathfrak{g})$, the graded algebra $\operatorname{gr} U(\mathfrak{g})$ is commutative.

Let $S(V)$ be a *symmetric algebra* for a finite-dimensional real linear space V, i.e., a free commutative algebra over the field $\mathbb{R}$, generated by elements of any basis of V. This algebra is graded by its subspaces $S_i(V)$, $i \geqslant 1$ consisting of homogeneous polynomials with their degrees equal to i.

For an element $Y \in \mathfrak{g}$ we denote by Y^* the corresponding element from $S(\mathfrak{g})$. The adjoint action of the group G on $\mathfrak{g}$ can be naturally extended to the action of G on the algebra $U(\mathfrak{g})$ according to the formula:

$$\mathrm{Ad}_q : Y_1 \cdot \ldots \cdot Y_i \to \mathrm{Ad}_q(Y_1) \cdot \ldots \cdot \mathrm{Ad}_q(Y_i),\; Y_1, \ldots, Y_i \in \mathfrak{g}\ ,$$

and similarly to the action of G on $S(\mathfrak{g})$.

The following theorem is well known [28, 41, 142].

Theorem 2.1 (Poincaré-Birgoff-Witt). *The commutative algebras* $\operatorname{gr} U(\mathfrak{g})$ *and* $S(\mathfrak{g})$ *are isomorphic. The set* $(e_{i_1} \cdot \ldots \cdot e_{i_k} \mid 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant N;\, k \in \mathbb{N})$ *is a base in* $U(\mathfrak{g})$.

Proposition 2.1 ([41], 2.4.5, 2.4.6). *The linear symmetrization map* $\lambda : S(\mathfrak{g}) \to U(\mathfrak{g})$ *, defined on monomials by the formula*

$$\lambda(e^*_{i_1} \cdot \ldots \cdot e^*_{i_k}) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} e_{i_{\sigma(1)}} \cdot \ldots \cdot e_{i_{\sigma(k)}}\ ,$$

where $\mathfrak{S}_k$ *is the group consisting of all permutations of* k *elements, is an isomorphism of linear spaces.*

For any subgroup K' of the group G denote by $U(\mathfrak{g})^{K'}$ and $S(\mathfrak{g})^{K'}$ the sets of all $\mathrm{Ad}_{K'}$-invariant elements in $U(\mathfrak{g})$ and $S(\mathfrak{g})$ respectively. Proposition 2.1 and the evident equality $\lambda \circ \mathrm{Ad}_q = \mathrm{Ad}_q \circ \lambda$, $\forall q \in G$ implies:

$$\lambda\left(S(\mathfrak{g})^{K'}\right) = U(\mathfrak{g})^{K'}. \tag{2.4}$$

We can consider vector fields on G as differential operators of the first order and any differential operator on the group G can be expressed as a polynomial in X^l_i or in X^r_i, $i = 1, \ldots, N$ with nonconstant coefficients. It is clear that every polynomial combination of left invariant vector fields on G with constant coefficients is an element from $\mathrm{LDiff}(G)$. This correspondence defines the homomorphism $\iota : U(\mathfrak{g}) \to \mathrm{LDiff}(G)$. Recall that everywhere by a polynomial on noncommutative arguments we mean an ordered one, i.e., each its monomial is an ordered product.

[2] We do not include the basic field $\mathbb{R}$ into $U(\mathfrak{g})$ as did some authors [41, 158], who put $U_0(\mathfrak{g}) := \mathbb{R}$, $[U_0(\mathfrak{g}), U(\mathfrak{g})] = 0$.

Theorem 2.2. *The homomorphism* $\iota : U(\mathfrak{g}) \to \mathrm{LDiff}(G)$ *is an isomorphism.*

Proof. We should only prove that the map ι is surjective. Let $D \in \mathrm{LDiff}(G)$, then there is the polynomial P such that

$$(Df)(e) = P(\partial/\partial t^1, \dots, \partial/\partial t^N) f\left(\exp(t^1 e_1 + \dots + t^N e_N)\right), \; f \in C^\infty(G) ,$$

since the set $(t^1, \dots t^N) \in \mathbb{R}^N$ is *normal* coordinates of the point $\exp(t^1 e_1 + \dots + t^N e_N)$ in some neighborhood of the unit element $e \in G$. Then due to the left invariance of D

$$(Df)(q) = P(\partial/\partial t^1, \dots, \partial/\partial t^N) f\left(q \exp(t^1 e_1 + \dots + t^N e_N)\right), \; q \in G .$$

Let

$$D_* = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} P\left(e_{\sigma_1}, \dots, e_{\sigma_N}\right) \in U(\mathfrak{g}) .$$

We claim that $\iota(D_*) = D$. For $X = t^i e_i \equiv \sum_{i=1}^N t^i e_i \in \mathfrak{g}$ one has

$$\begin{aligned}\left(t^i X_i^l\right)^k f(q) &= \left(X^l\right)^k f(q) = \left.\frac{d^k}{dt^k} f\left(g \exp tX\right)\right|_{t=0} \\ &= \left.\left(t^j \frac{\partial}{\partial y^j}\right)^k f\left(q \exp(y^i e_i)\right)\right|_{y^i=0} ,\end{aligned}$$

where $k \in \mathbb{N}$, $y^i := t t^i$. Comparing the terms proportional to the monomial $\prod_{i=1}^k t^{j_i}$, where $1 \leqslant j_i \leqslant N$, in both sides of the last equality, we get

$$\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} X^l_{j_{\sigma(1)}} \circ \dots \circ X^l_{j_{\sigma(k)}} f(q) = \left.\left(\prod_{i=1}^k \frac{\partial}{\partial y^{j_i}}\right) f(q \exp(y^j e_j))\right|_{y^j=0} .$$

This means that an operator $\Box \in \mathrm{LDiff}(G)$ of the form

$$\Box : \; f(q) \to \left.\left(\prod_{i=1}^k \frac{\partial}{\partial y^{j_i}}\right) f(q \exp(y^j e_j))\right|_{y^j=0}$$

lies in $\iota(U(\mathfrak{g}))$. By linearity we see that $\iota(D_*) = D$ and $\mathrm{LDiff}(G) = \iota(U(\mathfrak{g}))$. □

From this proof and Proposition 2.1 one gets also:

Corollary 2.1. *The composition* $\lambda^* := \iota \circ \lambda$ *is an isomorphism of linear spaces* $S(\mathfrak{g}) \to \mathrm{LDiff}(G)$ *and the following formula holds:*

$$\begin{aligned}\left(\lambda^*(P(Y^*_{i_1}, \dots, Y^*_{i_k}))f\right)(q) &= P(\partial/\partial t^{i_1}, \dots, \partial/\partial t^{i_k}) \\ &\quad \times f\left(q \exp(t^{i_1} Y_{i_1} + \dots + t^{i_k} Y_{i_k})\right)\big|_{t=0} \\ &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} P\left(Y^l_{i_{\sigma(1)}}, \dots, Y^l_{i_{\sigma(k)}}\right) f(q) ,\end{aligned} \tag{2.5}$$

where $Y_{i_j} \in \mathfrak{g}$, $j = 1, \dots, k$ *and* P *is a polynomial.*

For an operator $\square \in \mathrm{LDiff}(G)$ we call the element $(\lambda^*)^{-1}\square \in S(\mathfrak{g})$ the *lie-symbol* of $\square$. Let $\widetilde{\mathrm{Ad}}_q\square := \tilde{L}_q \circ \tilde{R}_{q^{-1}}(\square)$, $\square \in \mathrm{Diff}(G)$, $q \in G$; then clearly, $\widetilde{\mathrm{Ad}}_q$ is an automorphism of algebras $\mathrm{Diff}(G)$, $\mathrm{LDiff}(G)$, $\mathrm{RDiff}(G)$. Obviously,

$$\widetilde{\mathrm{Ad}}_q\square = \widetilde{R}_{q^{-1}}(\square),\ \square \in \mathrm{LDiff}(G)\ .$$

Lemma 2.1. *For an arbitrary element $q \in G$ it holds the following identity $\widetilde{\mathrm{Ad}}_q \circ \iota = \iota \circ \mathrm{Ad}_q$ of two maps from $U(\mathfrak{g})$ to $\mathrm{LDiff}(G)$.*

Proof. For $X \in \mathfrak{g}$ and $f \in C^\infty(G)$ one has

$$\begin{aligned}
&\widetilde{\mathrm{Ad}}_q \circ \iota(X)(f)\Big|_g = \widetilde{L}_q \circ \widetilde{R}_{q^{-1}} \circ \iota(X)(f)\Big|_g = \widetilde{R}_{q^{-1}} \circ \iota(X)(f)\Big|_g \\
&= \widehat{R}_{q^{-1}} \circ \iota(X)\left(\widehat{R}_q f\right)\Big|_g = \frac{d}{dt} f\left(gq\exp(tX)q^{-1}\right)\Big|_{t=0} \\
&= \frac{d}{dt} f\left(g\exp\left(t\,\mathrm{Ad}_q X\right)\right)\Big|_{t=0} = \left(\iota \circ \mathrm{Ad}_q X(f)\right)(g);\ g, q \in G\ .
\end{aligned}$$

Due to Theorem 2.2 elements $\iota(X)$, $X \in \mathfrak{g}$ generate $\mathrm{LDiff}(G)$, therefore it holds $\widetilde{\mathrm{Ad}}_q \circ \iota = \iota \circ \mathrm{Ad}_q$. $\square$

Proposition 2.2. $\mathrm{Z}\,\mathrm{LDiff}(G) = \lambda^*(S(\mathfrak{g})^G)$.

Proof. Since algebras $\mathrm{LDiff}(G)$ and $U(\mathfrak{g})$ are isomorphic, it holds $\iota(\mathrm{Z}\,U(\mathfrak{g})) = \mathrm{Z}\,\mathrm{LDiff}(G)$.

The center $\mathrm{Z}\,U(\mathfrak{g})$ coincides with the set $U(\mathfrak{g})^G$. Indeed, if $Y \in U(\mathfrak{g})^G$, then $[X, Y] = \frac{d}{dt}\mathrm{Ad}_{\exp(tX)} Y\big|_{t=0} = 0$, $\forall X \in \mathfrak{g}$ and therefore $[A, Y] = 0$, $\forall A \in U(\mathfrak{g})$. Conversely, if $Y \in \mathrm{Z}\,U(\mathfrak{g})$, then

$$\frac{d}{dt}\mathrm{Ad}_{\exp(tX)} Y = \frac{d}{dt}\mathrm{Ad}_{\exp(tX)} \circ \frac{d}{ds}\mathrm{Ad}_{\exp(sX)} Y\Big|_{s=0} = \mathrm{Ad}_{\exp(tX)}[X, Y] = 0\ ,$$

$\forall X \in \mathfrak{g}$, $\forall t \in \mathbb{R}$. Therefore $\mathrm{Ad}_{\exp X} Y = Y$, $\forall X \in \mathfrak{g}$ and $\mathrm{Ad}_q Y = Y$, $\forall q \in G$, since a connected Lie group is generated by any neighborhood of its unit element.

Now the proposition follows from (2.4). $\square$

The Killing form $\mathrm{Kil}_{\mathfrak{g}'}(X, Y)$ on any Lie algebra $\mathfrak{g}'$ is defined as the trace of the map $Z \to [X, [Y, Z]]$, $X, Y, Z \in \mathfrak{g}'$.

Suppose that the Lie algebra $\mathfrak{g}$ is semisimple, then its Killing form $\mathrm{Kil}_{\mathfrak{g}}$ is nondegenerate. Let e_i, $i = 1, \ldots, N$ be a base in $\mathfrak{g}$ and e^j, $j = 1, \ldots, N$ be the dual base such that $\mathrm{Kil}_{\mathfrak{g}}(e_i, e^j) = \delta_i^j$.

Lemma 2.2. *The element $C := e_i e^i \in U(\mathfrak{g})$ does not depend on the choice of the base e_i and lies in $\mathrm{Z}\,U(\mathfrak{g})$. Also, it holds $C = e^i e_i$. Thus $\iota(C) \in \mathrm{Z}\,\mathrm{LDiff}(G)$.*

Proof. If $\bar{e}_i = a_i^j e_j$ is another base in $\mathfrak{g}$, then its dual base is $\bar{e}^j = b_i^j e^i$, where $b_k^i a_j^k = \delta_j^i$. One obtains that $\bar{e}_i\bar{e}^i = a_i^j b_k^i e_j e^k = \delta_k^j e_j e^k = e_j e^j$. Similarly, $e^i e_i = \bar{e}^i \bar{e}_i$.

The Killing form is $\mathrm{ad}_{\mathfrak{g}}$-invariant. Therefore

$$\alpha_i^j(X) := \mathrm{Kil}_{\mathfrak{g}}([X, e_i], e^j) = -\mathrm{Kil}_{\mathfrak{g}}(e_i, [X, e^j]) =: -\beta_i^j(X),\ \forall X \in \mathfrak{g}\ ,$$

where $[X, e_i] = \alpha_i^j(X)e_j$, $[X, e^i] = \beta_j^i(X)e^j$. Thus

$$\begin{aligned}[X, C] = [X, e_i]e^i + e_i[X, e^i] &= \alpha_i^j(X)e_j e^i + \beta_j^i(X)e_i e^j \\ &= \left(\alpha_i^j(X) + \beta_i^j(X)\right) e_j e^i = 0\ .\end{aligned}$$

It is well known that the base e_i can be chosen in such a way that $e_i = \pm e^i$. In this case one gets the equality $C = e^i e_i = e_i e^i$, which is valid therefore for an arbitrary base due to the first claim of the lemma. □

In fact, the element C was introduced by H. Casimir in [33]. Its image under any representation of $\mathfrak{g}$ is called a *Casimir operator*.

2.1.3 Invariant Differential Operators on a Homogeneous Spaces

Functions on the G-homogeneous space $M \cong G/K$ are in one-to-one correspondence with functions on the group G that are invariant under the right action of the subgroup K. This correspondence is defined by the formula $\zeta : f \to \tilde{f} := f \circ \pi_1$, where f is a function on the space M, $\tilde{f}$ is the corresponding function on the group G, and as before π_1 is the canonical projection $G \to G/K$. If f is smooth, then so is $\tilde{f}$. Define a map

$$\eta : \mathrm{LDiff}_K(G) \to \mathrm{Diff}_G(M)$$

by the formula

$$\eta(\Box)f = \zeta^{-1} \circ \Box \circ \zeta(f),\ f \in C^\infty(M),\ \Box \in \mathrm{LDiff}_K(G)\ .$$

This map is well defined, since the function $\Box \circ \zeta(f)$ is right-invariant with respect to the subgroup K. Evidently, the map η is a homomorphism.

Assume now that the space M is *reductive*, i.e., $\mathrm{Ad}_K\,\mathfrak{p} \subset \mathfrak{p}$ and then $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$.

This assumption will be valid in the case when M is a Riemannian space and G is its isometry group or a subgroup of this group. Indeed, in this case the stationary subgroup K is compact, since it is also the subgroup of the group $\mathrm{SO}(m)$. By the group averaging on K one can define a Ad_K-invariant scalar product on $\mathfrak{g}$ and choose the subspace $\mathfrak{p}$ orthogonal to $\mathfrak{k}$ with respect to this product [66, 92]. This case is the only one which we consider in the following chapters in connection with two-point homogeneous spaces.

In some neighborhood of the point $x_0 \in M$ one can define coordinates $(x^1, \ldots, x^m)$, of the point $\pi_1(\exp(\sum_{i=1}^m x^i e_i))$. Let $P\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\right)$ be a polynomial expression of an operator $\Box \in \mathrm{Diff}_G(M)$ at the point x_0. Define a map:

$$\varkappa : \mathrm{Diff}_G(M) \to S(\mathfrak{p}) \subset S(\mathfrak{g})\ , \tag{2.6}$$

by the formula $\varkappa(\Box) = P(e_1^*, \dots, e_m^*) \in S(\mathfrak{p})$. By direct calculations we see that the map $\varkappa$ doesn't depend on the choice of the basis in the space $\mathfrak{p}$. We call the element $\varkappa(\Box)$ the *lie-symbol* of the operator $\Box \in \mathrm{Diff}_G(M)$. In the case $M = G$ one gets the same lie-symbol as in the previous section.

Since $\hat\tau_{q^{-1}} \circ \Box \circ \hat\tau_q = \Box$, $\forall q \in G$, for any $f \in C^\infty(M)$ one has

$$\begin{aligned}
&(\Box f)(qx_0) = \hat\tau_{q^{-1}} \circ \Box(f)(x_0) = \Box \circ \hat\tau_{q^{-1}}(f)(x_0) \\
&= P\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right) f\left(\pi\left(q \exp\Big(\sum_{i=1}^m x^i e_i\Big)\right)\right)\Bigg|_{x^i=0} \qquad (2.7)\\
&= P\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right) \tilde f\left(q \exp\Big(\sum_{i=1}^m x^i e_i\Big)\right)\Bigg|_{x^i=0} = \eta \circ \lambda^*(P(e_1^*, \dots, e_m^*)) f(qx_0) \,.
\end{aligned}$$

The last equality follows from (2.5). Consider $q \in K$. Let $\mathrm{Ad}_q\, e_i =: A_i^j e_j$ and $y^j := A_i^j x^i$. Then $\frac{\partial}{\partial x^i} = A_i^j \frac{\partial}{\partial y^j}$, $\mathrm{Ad}_q\, e_i^* = A_i^j e_j^*$ and due to $qx_0 = x_0$ it holds

$$\begin{aligned}
(\Box f)(x_0) &= P\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right) \tilde f\left(\exp\left(\sum_{i=1}^m x^i\, \mathrm{Ad}_q\, e_i\right)\right)\Bigg|_{x^i=0} \\
&= \widetilde P\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m}\right) \tilde f\left(\exp\Big(\sum_{i=1}^m y^i e_i\Big)\right)\Bigg|_{y^i=0} \,,
\end{aligned}$$

where $\widetilde P(e_1^*, \dots, e_m^*) = P\left(\mathrm{Ad}_q\, e_1^*, \dots, \mathrm{Ad}_q\, e_m^*\right) = \mathrm{Ad}_q\, P(e_1^*, \dots, e_m^*)$, since the map $\varkappa$ does not depend on a choice of the basis for the space $\mathfrak{p}$ and, in particular, it is not changed by the transition to the basis $\mathrm{Ad}_q\, e_i$. On the other hand, polynomials P and $\widetilde P$ are two expressions of the operator $\Box$ at the point x_0, so $\widetilde P(e_1^*, \dots, e_m^*) = P(e_1^*, \dots, e_m^*)$, i.e., $P(e_1^*, \dots, e_m^*) \in S(\mathfrak{p})^K$, where $S(\mathfrak{p})^K$ is the set of all Ad_K-invariant elements in $S(\mathfrak{p})$. Note that $S(\mathfrak{p})^K \subset S(\mathfrak{g})^K$.

Reasoning in the reverse order, we get from (2.7), that if $P(e_1^*, \dots, e_n^*) \in S(\mathfrak{p})^K$, then $\lambda^*(P(e_1^*, \dots, e_n^*)) \in \mathrm{LDiff}_K(G)$ and $\eta \circ \lambda^*(P(e_1^*, \dots, e_n^*)) \in \mathrm{Diff}_G(M)$. Also, the expression of the operator $\eta \circ \lambda^*(P(e_1^*, \dots, e_n^*))$ at the point x_0 through coordinates $x^1, \dots, x^m$ is again $P\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right)$, therefore $\varkappa \circ \eta \circ \lambda^* = \mathrm{id}$.

The formula (2.7) implies $\eta \circ \lambda^* \circ \varkappa = \mathrm{id}$, i.e., the following diagram is commutative:

$$\begin{array}{ccccc}
S(\mathfrak{p})^K & \xrightarrow{\lambda} & U(\mathfrak{g})^K & \xrightarrow{\iota} & \mathrm{LDiff}_K(G) \\
\nwarrow \varkappa & & & & \swarrow \eta \\
& \mathrm{Diff}_G(M) & \xrightarrow{\mathrm{id}} & \mathrm{Diff}_G(M) &
\end{array}$$

Hence the maps $\varkappa$ is bijective, the map η is surjective and the map $\lambda^* = \iota \circ \lambda$ is injective that implies

$$\mathrm{LDiff}_K(G) = \lambda^*(S(\mathfrak{p})^K) \oplus \ker \eta \,. \qquad (2.8)$$

Lemma 2.3. *For any Lie algebra* $\mathfrak{g}'$ *and any its subalgebra* $\mathfrak{k}'$ *denote by* $U(\mathfrak{g}')\mathfrak{k}'$ *the left ideal in* $U(\mathfrak{g}')$ *generated by* $\mathfrak{k}'$. *Let* $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$ *be an expansion of* $\mathfrak{g}'$ *into a direct sum of linear subspaces. Then the linear space* $U(\mathfrak{g}')$ *admits the expansion:*

$$U(\mathfrak{g}') = U(\mathfrak{g}')\mathfrak{k}' \oplus \lambda(S(\mathfrak{p}')) \ . \tag{2.9}$$

Proof. The proof is by the straightforward induction with respect to the degree of a polynomial expression of an element from $U(\mathfrak{g}')$ through elements of $\mathfrak{g}'$. □

Let $(U(\mathfrak{g})\mathfrak{k})^K$ be the set of all Ad_K-invariant elements in $U(\mathfrak{g})\mathfrak{k}$. It is a two-sided ideal in $U(\mathfrak{g})^K$, since for elements $f \in \mathfrak{k}$ and $g \in U(\mathfrak{g})^K$ one has $fg = \mathrm{ad}_f\, g \ + \ gf = gf$. Thus, the factor algebra $U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ is well defined. If $\mathfrak{k}' = \mathfrak{k}$, $\mathfrak{p}' = \mathfrak{p}$, then both summand in (2.9) are Ad_K-invariant and we have

$$U(\mathfrak{g})^K = (U(\mathfrak{g})\mathfrak{k})^K \oplus \lambda(S(\mathfrak{p})^K) \ . \tag{2.10}$$

According to Theorem 2.2 the map $\iota : U(\mathfrak{g}) \to \mathrm{LDiff}(G)$ is an isomorphism. Therefore, the Lemma 2.1 implies that the map $\iota|_{U(\mathfrak{g})^K}$ is an isomorphism between algebras $U(\mathfrak{g})^K$ and $\mathrm{LDiff}_K(G)$. From (2.8) and (2.10) one gets

$$\mathrm{LDiff}_K(G) = \iota\left(U(\mathfrak{g})^K\right) = \iota\left((U(\mathfrak{g})\mathfrak{k})^K\right) \oplus \lambda^*(S(\mathfrak{p})^K) \ . \tag{2.11}$$

Clearly $\eta \circ \iota\left((U(\mathfrak{g})\mathfrak{k})^K\right) = 0$ and comparing (2.11) with (2.8) we obtain $\iota\left((U(\mathfrak{g})\mathfrak{k})^K\right) = \ker \eta$.

The considerations above can be concluded in the following theorem:

Theorem 2.3. *The homomorphism* $\eta \circ \iota$ *induces the isomorphism*

$$\iota^* : U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K \to \mathrm{Diff}_G(M) \ .$$

The map $\eta \circ \lambda^* = \eta \circ \iota \circ \lambda$ *is the linear bijection of linear spaces* $S(\mathfrak{p})^K$ *and* $\mathrm{Diff}_G(M)$.

The filtration of the algebra $U(\mathfrak{g})$ *induces the filtration of the algebra* $\mathrm{Diff}_G(M)$, *which coincides with the natural filtration of* $\mathrm{Diff}_G(M)$ *as the algebra of differential operators.*

Since $\mathrm{Z\,LDiff}(G) \subset \mathrm{LDiff}_K(G)$, from Proposition 2.2 and Lemma 2.2 one gets

Corollary 2.2. *The set* $\eta \circ \lambda^*(S(\mathfrak{g})^G)$ *and the element* $\eta \circ \iota(C)$ *lie in* $\mathrm{Z\,Diff}_G(M)$.

This means that every Ad_G-invariant element from $S(\mathfrak{g})$ induces the element from $\mathrm{Z\,Diff}_G(M)$. The operator $\eta \circ \iota(C)$ is the Casimir one.

Remark 2.1. *We shall use an operator* $\lambda^* \circ \varkappa(\Box) \in \mathrm{LDiff}_K(G)$ *as a lift* $\widetilde{\Box}$ *of an operator* $\Box \in \mathrm{Diff}_G(M)$ *onto the group* G .

The following theorem gives a prescription for calculating this lift.

Theorem 2.4. *Let* $\Box|_{x_0} = P(\widetilde{Y}_1, \ldots, \widetilde{Y}_k)\Big|_{x_0}$ *, where* P *is a polynomial invariant with respect to any permutation of its arguments,* $Y_1, \ldots, Y_k \in \mathfrak{g}$ *and* $\widetilde{Y}_1, \ldots, \widetilde{Y}_k$ *are corresponding Killing vector fields. Then it holds* $\lambda^* \circ \varkappa(\Box) = P(Y_1^l, \ldots, Y_k^l)$ *. This formula for the lift depends on the expansion* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \equiv \mathfrak{k}_{x_0} \oplus \mathfrak{p}_{x_0}$ *.*

Proof. Since

$$\widetilde{Y}^k f(x_0) = \frac{d^k}{dt^k} f\left(\exp(tY)x_0\right)\Big|_{t=0}, \; Y \in \mathfrak{g}, \; f \in C^\infty(M)\,,$$

by the same arguments as in the proof of Theorem 2.2 we obtain for every polynomial $\widetilde{P}$ and elements $Y_1, \ldots, Y_k \in \mathfrak{g}$ the formula:

$$\begin{aligned} &\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \widetilde{P}\left(\widetilde{Y}_1, \ldots, \widetilde{Y}_k\right) f(x_0) \\ &= \widetilde{P}\left(\partial/\partial t^1, \ldots, \partial/\partial t^k\right) f\,\left(\exp(t^1 Y_1 + \ldots + t^k Y_k) x_0\right)\Big|_{t^i=0}\,. \end{aligned}$$

This yields

$$\begin{aligned} \Box f(x_0) &= P\left(\widetilde{Y}_1, \ldots, \widetilde{Y}_k\right) f(x_0) \\ &= P\left(\partial/\partial t^1, \ldots, \partial/\partial t^k\right) f\,\left(\exp(t^1 Y_1 + \ldots + t^k Y_k) x_0\right)\Big|_{t^i=0}\,. \end{aligned}$$

Thus, $\varkappa(\Box) = P\left(Y_1^*, \ldots, Y_k^*\right)$ and, due to corollary 2.1 and the symmetry of P, one gets: $\lambda^* \circ \varkappa(\Box) = P\left(Y_1^l, \ldots, Y_k^l\right)$. □

The theoretical base for calculation invariants in Chap. 3 is the following proposition:

Proposition 2.3 ([201] Chap. 11, theorems 3, 4). *Let* G *be a compact Lie group, acting in a real or complex vector space* V*. Then the algebra* $S(V^*)^G$ *of its polynomial invariants is finitely generated. If the space* V *is real, then orbits of the* G*-action are separated by polynomial invariants.*

In the search of invariant elements we shall use also the following obvious lemma.

Lemma 2.4. *Let a group* G *acts in Euclidean space* V, $\dim V = m$ *by orthogonal transformations. Let* e_i, $i = 1, \ldots m$ *be an orthonormal base and* x^i, $i = 1, \ldots m$ *be corresponding coordinates. Let* $P(x^1, \ldots, x^m)$ *be a polynomial invariant of* G*-action in* V*. Then* $P(e_1, \ldots, e_m)$ *is a* G*-invariant element from* $S(V)$ *and this is a one-to-one correspondence between polynomial invariants in* V *and invariant elements in* $S(V)$*.*

Proof. If A_g is a matrix of an action of $g \in G$ in the base $x^1, \ldots, x^m$ of the space V^*, then A_g^T is the matrix of g-action in the base $e_1, \ldots, e_m$. Due to the orthogonality of G-action in V one has $A_g^T = A_g^{-1} = A_{g^{-1}}$. This completes the proof. □

2.1.4 Representation of the Algebra $\mathrm{Diff}_G(M)$ by Generators and Relations

Let

$$P(a_1, \ldots, a_k) = \sum_l \sum_{(i_1,\ldots,i_l)} p_{(i_1,\ldots,i_l)} a_{i_1} \cdot \ldots \cdot a_{i_l} \ , \qquad (2.12)$$

be an ordered polynomial, where $(i_1, \ldots i_l)$ are selections from the set $(1, \ldots, k)$, a_i are elements of some associative algebra $\mathcal{A}$ over a field $\mathbb{K}$, and $p_{(i_1,\ldots,i_l)} \in \mathbb{K}$. Call polynomial (2.12) *symmetric* if all its coefficients $p_{(i_1,\ldots,i_l)}$ satisfy the equality:

$$p_{(i_1,\ldots,i_l)} = p_{(\sigma(i_1),\ldots\sigma(i_l))} \ , \ \forall \sigma \in \mathfrak{S}_l \ .$$

Let $\pi_2 : U(\mathfrak{g})^K \to U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ be the canonical projection. One can get the relations in $\mathrm{Diff}_G(M) \cong U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ operating in $U(\mathfrak{g})^K$ modulo $(U(\mathfrak{g})\mathfrak{k})^K$. This approach leads to simpler calculations than the operations through local coordinates on M (like in [146]), which require quite cumbersome calculation even in the relatively simple case of $M = \mathbf{H}^n(\mathbb{R})_{\mathbf{S}}$.

Let (g_i) be a set of generators[3] of the commutative algebra $S(\mathfrak{p})^K \subset S(\mathfrak{p})$. Without loss of generality one can suppose that all g_i are homogeneous elements w.r.t. the grading of $S(\mathfrak{p})$. Then due to the expansion (2.10) the elements $\pi_2 \circ \lambda(g_i)$ generate the algebra $U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$.

Relations for the elements $\pi_2 \circ \lambda(g_i)$ are of two types. First type consists of relations induced by relations in $U(\mathfrak{g})$. Due to the universality of $U(\mathfrak{g})$ all these relation are commutator ones, induced by the Lie operation in $\mathfrak{g}$. They are reduced to *commutator relations* or *relations of the first type* of the simplest form: $[D_1, D_2] = \widetilde{D}$, where the operators $D_1, D_2 \in U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ have degrees m_1 and m_2 respectively and the degree of $\widetilde{D} \in U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ is less or equal $m_1 + m_2 - 1$.

Suppose now that there is a relation in $U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ of the form

$$P(\pi_2 \circ \lambda(g_1), \ldots, \pi_2 \circ \lambda(g_k)) = 0$$

or equivalently in $U(\mathfrak{g})$ of the form

$$P(\lambda(g_1), \ldots, \lambda(g_k)) = \widetilde{D} \ , \qquad (2.13)$$

where $P(\lambda(g_1), \ldots, \lambda(g_k))$ is an ordered polynomial and $\widetilde{D} \in (U(\mathfrak{g})\mathfrak{k})^K$. Using the commutator relations for $\lambda(g_i)$, $i = 1, \ldots, k$, one can reduce the polynomial $P(\lambda(g_1), \ldots, \lambda(g_k))$ to a symmetric polynomial $P_s(\lambda(g_1), \ldots, \lambda(g_k))$ and (2.13) becomes:

$$P_s(\lambda(g_1), \ldots, \lambda(g_k)) = \sum_l \sum_{(i_1,\ldots,i_l)} p_{(i_1,\ldots,i_l)} \lambda(g_{i_1}) \cdot \ldots \cdot \lambda(g_{i_l}) = D_* \ , \quad (2.14)$$

where $(i_1, \ldots i_l)$ are selections from the set $(1, \ldots, k)$ and $D_* \in (U(\mathfrak{g})\mathfrak{k})^K$. Relation (2.14) may be trivial:

[3] There could be some polynomial relations or *syzygies* between them. Relations between syzygies, if exist, are called syzygies of the second order and so on.

$$p_{(i_1,\ldots,i_l)} = 0,\ \forall(i_1,\ldots,i_l),\ D_* = 0\ .$$

This means that (2.13) is a commutator relation. Suppose that relation (2.14) is nontrivial. Let $P_1(\lambda(g_1),\ldots,\lambda(g_k))$ be the sum of monomials from the polynomial $P_s(\lambda(g_1),\ldots,\lambda(g_k))$ with the highest total degree in $U(\mathfrak{g})$. Consider the polynomial $P_1(t_1,\ldots,t_k)$ with commutative variables $t_1,\ldots,t_k$. Due to the definition of the symmetric polynomial (2.14), the polynomial $P_1(t_1,\ldots,t_k)$ is also nontrivial. On the other hand from (2.14) one gets that $P_1(g_1,\ldots,g_k) = 0$ due to the expansion $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Therefore any relation for generators $\pi_2 \circ \lambda(g_1),\ldots,\pi_2 \circ \lambda(g_k)$ of the algebra $U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K$ modulo commutator relations and relations of lower degrees corresponds to the relation for homogeneous generators $g_1,\ldots,g_k$ of the commutative algebra $S(\mathfrak{p})^K$. We call such relations *the relations of the second type.*

Conversely, let

$$P_1(g_1,\ldots,g_k) = 0$$

be a nontrivial relation in the algebra $S(\mathfrak{p})^K$. Without loss of generality one can suppose that the polynomial $P_1(g_1,\ldots,g_k)$ is homogeneous (i.e., all its monomials have the same degree d in $S(\mathfrak{p})$) and ordered. Then it is evident that

$$P_1(\lambda(g_1),\ldots,\lambda(g_k)) = \widetilde{D}\ , \tag{2.15}$$

where $\widetilde{D}$ is an element from $U(\mathfrak{g})^K$ of a degree less than d. After reducing relation (2.15) modulo $(U(\mathfrak{g})\mathfrak{k})^K$ one gets the relation in $U(\mathfrak{g})^K/\,(U(\mathfrak{g})\mathfrak{k})^K$:

$$P(\pi_2 \circ \lambda(g_1),\ldots,\pi_2 \circ \lambda(g_k)) = 0\ ,$$

where P is a polynomial of the degree d, coinciding in leading terms with the polynomial P_1.

Thus, one gets

Proposition 2.4. *There is a one-to-one correspondence described above between relations for generators $g_1,\ldots,g_k$ in the algebra $S(\mathfrak{p})^K$ and relations for generators $\pi_2 \circ \lambda(g_1),\ldots,\pi_2 \circ \lambda(g_k)$ in the algebra $U(\mathfrak{g})^K/\,(U(\mathfrak{g})\mathfrak{k})^K$ modulo relations of lower degree and commutator relations.*

For simplicity throughout the whole book we consider invariance of differential operators only w.r.t. the identity component of a whole isometry group.

There exists a unique (up to a constant factor) left- (or right-) invariant measure on any Lie group (the Haar measure [66, 88]). Denote by μ_G some left-invariant Haar measure on G. A measure on the space M, generated by a G-invariant metric is also G-invariant. All G-invariant measures on M are proportional [66] and one will define such a measure if one puts $\mu_M(V) = \mu_G(\pi_1^{-1}(V))$ for any compact set $V \in M$. The group K_{x_0} is compact, so the set $\pi_1^{-1}(V)$ is also compact and $\mu_G(\pi_1^{-1}(V)) < \infty$. The measure μ_M is left-invariant, since the measure μ_G is left-invariant.

Left-invariant measures on all unimodular groups and particulary on compact groups are also right-invariant. The change of the point $x_0 \in M$ by $x_1 = qx_0, q \in G$ leads to the change of the pullback $\pi_1^{-1}(V)$ by $\pi_1^{-1}(V)q^{-1}$, while identifying M with G/K_{x_0}. Therefore, the G-invariant measure μ_M for the unimodular group G does not depend on the choice of x_0.

2.2 Laplace–Beltrami Operator in a Moving Frame

In this section we do not regard M as a homogeneous manifold with respect to the isometry group until the homogeneity is declared explicitly. Here we shall find the polynomial P from Theorem 2.4, corresponding to the Laplace–Beltrami operator on a Riemannian space M. First, let us obtain the expression for the Laplace–Beltrami operator in arbitrary moving frame.

Denote the metric on M by g. Let $x^i, i = 1, \ldots, m$ be local coordinates in a domain $U \subset M$ and $g_{ij}dx^i dx^j$ be the corresponding expression of the metric g on U. The range for all indices in this section is $1, \ldots, m$. The Laplace–Beltrami operator generated by the metric g has the following form on U:

$$\triangle_g = (\gamma)^{-1/2} \frac{\partial}{\partial x^i} \left(\sqrt{\gamma} g^{ij} \frac{\partial}{\partial x^j} \right) , \tag{2.16}$$

where $\gamma = |\det g_{ij}|$ and $g^{ij}(x)$ is the inverse of the matrix $g_{ij}(x)$. The operator $\triangle_g$ is conserved by all isometries of the space M. Conversely, if the operator $\triangle_g$ is conserved by some diffeomorphism of the space M, then this diffeomorphism is an isometry [66].

Let $\xi_i = \phi_i^k(x)\partial/\partial x^k$ be vector fields on U forming a moving frame. Any vector field is a differential operator of the first order. Using a composition one can express any differential operator on U via this moving frame with some (generally nonconstant) coefficients. Let $\|\psi_j^i\|$ be the inverse for the matrix $\|\phi_i^k\|$. Then $\partial/\partial x^k = \psi_k^m \xi_m$ and $\hat{g}_{ij} := g(\xi_i, \xi_j) = \phi_i^k \phi_j^m g_{km}$ are the coefficients of the metric g with respect to the moving frame ξ_i. This implies that $\hat{g}^{ij} = \psi_k^i g^{kn} \psi_n^j$ and $\hat{\gamma} := |\det \hat{g}_{ij}| = \phi^2\gamma$, where $\phi = \det \|\phi_i^k\|$. Substituting these formulae in (2.16), one gets:

$$\begin{aligned}
\triangle_g &= (\hat{\gamma})^{-1/2} \phi \psi_i^q \xi_q \circ \left(\phi^{-1} \hat{\gamma}^{1/2} \phi_k^i \hat{g}^{kn} \phi_n^j \psi_j^p \xi_p \right) \\
&= (\hat{\gamma})^{-1/2} \phi \psi_i^q \xi_q \circ \left(\phi^{-1} \hat{\gamma}^{1/2} \phi_k^i \hat{g}^{kn} \xi_n \right) \\
&= (\hat{\gamma})^{-1/2} \psi_i^q \phi_k^i \xi_q \circ \left(\hat{\gamma}^{1/2} \hat{g}^{kn} \xi_n \right) + \phi \psi_i^q \hat{g}^{kn} \xi_q \left(\phi^{-1} \phi_k^i \right) \xi_n \\
&= (\hat{\gamma})^{-1/2} \xi_k \circ \left(\hat{\gamma}^{1/2} \hat{g}^{kn} \xi_n \right) + \hat{g}^{kn} L_k \xi_n ,
\end{aligned}$$

where $L_k = \phi \psi_i^q \xi_q \left(\phi^{-1} \phi_k^i \right) = \psi_i^q \xi_q \left(\phi_k^i \right) + \phi \xi_k \left(\phi^{-1} \right) = \psi_i^q \xi_q \left(\phi_k^i \right) - \phi^{-1} \xi_k(\phi)$.

Denote $\Phi(x) = \|\phi_k^i(x)\|$. Using the well-known formula $\det \exp(A) = \exp(\mathrm{Tr} A)$, where A is an arbitrary complex matrix, we get:

$$\begin{aligned}
\phi^{-1} \xi_k(\phi) &= \xi_k(\ln \phi) = \xi_k \left(\ln \exp(\mathrm{Tr} \ln \Phi) \right) = \xi_k \left(\mathrm{Tr} \ln \Phi \right) \\
&= \mathrm{Tr} \left(\Phi^{-1} \xi_k(\Phi) \right) = \psi_i^q \xi_k(\phi_q^i) ,
\end{aligned} \tag{2.17}$$

since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any square matrices A and B. On the other hand, the following equations for commutators of vector fields

$$[\xi_i,\xi_j]=\xi_i\left(\phi_j^k\right)\frac{\partial}{\partial x^k}-\xi_j\left(\phi_i^k\right)\frac{\partial}{\partial x^k}=\left(\xi_i\left(\phi_j^k\right)-\xi_j\left(\phi_i^k\right)\right)\psi_k^q\xi_q=:c_{ij}^q\xi_q$$

define the functions

$$c_{ij}^q=\left(\xi_i\left(\phi_j^k\right)-\xi_j\left(\phi_i^k\right)\right)\psi_k^q$$

on U. So, in view of (2.17), one has:

$$L_k=\left(\xi_q\left(\phi_k^i\right)-\xi_k\left(\phi_q^i\right)\right)\psi_i^q=c_{qk}^q\ .$$

Thus, the formula for the Laplace–Beltrami operator in the moving frame ξ_i is:

$$\triangle_g=(\hat\gamma)^{-1/2}\,\xi_q\circ\left(\sqrt{\hat\gamma}\hat g^{qn}\xi_n\right)+\hat g^{kn}c_{qk}^q\xi_n\ . \tag{2.18}$$

Suppose now that ξ_i are Killing vector fields for the metric g in U.[4] Transform formula (2.18) to the form $\triangle_g=a^{ij}\xi_i\circ\xi_j+b^i\xi_i$. It is clear that $a^{ij}=\hat g^{ij}$ and one only has to find the coefficients b^i. It is well-known that

$$X\left(g(Y,Z)\right)=(\pounds_Xg)(Y,Z)+g([X,Y],Z)+g(Y,[X,Z]),\ X,Y,Z\in\mathcal X(M)\ , \tag{2.19}$$

where $\pounds_X$ is a Lie derivative with respect to a field X. Now formulae $\pounds_{\xi_k}g=0$, (2.17), (2.18) and (2.19) imply

$$\begin{aligned}b^i\hat g_{ij}&=\hat\gamma^{-1/2}\xi_k(\hat\gamma^{1/2}\hat g^{ki})\hat g_{ij}+\hat g^{ki}c_{qk}^q\hat g_{ij}=\xi_k(\hat g^{ki}\hat g_{ij})-\xi_k(\hat g_{ij})\hat g^{ki}+\frac{1}{2\hat\gamma}\xi_k(\hat\gamma)\hat g^{ki}\hat g_{ij}\\&+c_{qk}^q\delta_j^k=\xi_k(\delta_j^k)-\xi_k(g(\xi_i,\xi_j))\hat g^{ki}+\frac12\hat g^{qi}\xi_k(\hat g_{qi})\delta_j^k+c_{qj}^q=-g([\xi_k,\xi_i],\xi_j)\hat g^{ki}\\&-g(\xi_i,[\xi_k,\xi_j])\hat g^{ki}+\frac12\hat g^{qi}g([\xi_j,\xi_q],\xi_i)+\frac12\hat g^{qi}g(\xi_q,[\xi_j,\xi_i])+c_{qj}^q=-\hat g_{iq}c_{kj}^q\hat g^{ki}\\&+\frac12\hat g^{qi}c_{jq}^k\hat g_{ki}+\frac12\hat g^{qi}c_{ji}^k\hat g_{qk}+c_{qj}^q=-c_{qj}^q+\frac12c_{jq}^q+\frac12c_{jq}^q+c_{qj}^q=c_{jq}^q\ ,\end{aligned} \tag{2.20}$$

taking into account the antisymmetry of the tensor c_{ki}^q with respect to lower indices. Thus, one gets $b^i=c_{jq}^q\hat g^{ji}$. We can summarize this reasoning in the following proposition:

Proposition 2.5. *In the moving frame ξ_i, consisting of Killing vector fields, the Laplace–Beltrami operator has the following form*

$$\triangle_g=\sum_{i,j=1}^m\hat g^{ij}\xi_i\circ\xi_j+\sum_{i,j,q=1}^m c_{jq}^q\hat g^{ji}\xi_i\ .$$

If the space M is homogeneous and $\xi_i=\tilde e_i$, $i=1,\dots,m$ in notations of Sect. 2.1, *then theorem* 2.4 *implies that the lift of the operator $\triangle_g$ onto the group G has the form:*

$$\widetilde\triangle_g=\sum_{i,j=1}^m\hat g^{ij}\big|_{x_0}X_i^l\circ X_j^l+\sum_{i,j,q=1}^m c_{jq}^q\hat g^{ji}\big|_{x_0}X_i^l\ .$$

[4] One can find a moving frame, consisting of Killing vector fields, on any homogeneous Riemannian manifold.

Remark 2.2. *Sometimes vector fields ξ_i can be chosen in such a way that $c^q_{jq} = 0$. In this case one has $\triangle_g = \hat{g}^{ij}\xi_i \circ \xi_j$ and $\widetilde{\triangle}_g = \hat{g}^{ij}\big|_{x_0} X^l_i \circ X^l_j$.*

In the sequel, we shall use the expression for the Levi-Civita connection ∇ with respect to Killing vector fields:

Lemma 2.5 ([17], 7.27). *Let ξ_i, $i = 1, 2, 3$ be Killing vector fields. Then*

$$g(\nabla_{\xi_1}\xi_2, \xi_3) = \frac{1}{2}g(\xi_1, [\xi_2, \xi_3]) + \frac{1}{2}g(\xi_2, [\xi_1, \xi_3]) + \frac{1}{2}g([\xi_1, \xi_2], \xi_3) .$$

In particular,

$$g(\nabla_{\xi_1}\xi_1, \xi_3) = g(\xi_1, [\xi_1, \xi_3]) . \tag{2.21}$$

Proof. First, we shall prove that if X is a Killing vector field and Y, Z are arbitrary smooth vector fields, then

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0 .$$

In particular for $Z = Y$ or $Z = X$:

$$g(\nabla_Y X, Y) = -g(\nabla_Y X, Y) = 0,\ g(\nabla_Y X, X) = -g(\nabla_X X, Y) . \tag{2.22}$$

Indeed, it holds

$$\begin{aligned} 0 = (\pounds_X g)(Y, Z) &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_X Y, Z) + g(\nabla_Y X, Z) \\ &\quad - g(Y, \nabla_X Z) + g(Y, \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) , \end{aligned}$$

where we used the fact that the Levi-Civita connection ∇ is torsion free: $\nabla_Y Z - \nabla_Z Y = [Y, Z]$. Using (2.22) and the torsion free property again one gets:

$$g([\xi_1, \xi_3], \xi_1) = g(\nabla_{\xi_1}\xi_3 - \nabla_{\xi_3}\xi_1, \xi_1) = g(\nabla_{\xi_1}\xi_1, \xi_3) .$$

After polarization (i.e., using substitution $\xi_1 \to \xi_1 + \xi_2$)) the latter formula implies:

$$\begin{aligned} g([\xi_1, \xi_3], \xi_2) + g([\xi_2, \xi_3], \xi_1) &= g(\nabla_{\xi_1}\xi_2, \xi_3) + g(\nabla_{\xi_2}\xi_1, \xi_3) \\ &= 2g(\nabla_{\xi_1}\xi_2, \xi_3) + g([\xi_2, \xi_1], \xi_3) . \end{aligned}$$

□

2.3 Self-Adjointness of Hamiltonians

2.3.1 Self-Adjointness of Operators in Abstract Hilbert Spaces

According to Stone's theorem [144] a Hamiltonian H of a quantum mechanical system defines a quantum dynamic (a one parametric group of unitary transformations in a Hilbert space $\mathcal{H}$) if and only if H is a self-adjoint operator in $\mathcal{H}$. This section contains basic facts, connected with the notion of the self-adjointness (for details see [44, 144, 145]), which is used below for differential operators on Riemannian manifolds.

Let $\mathcal{H}$ be an unitary Hilbert space with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, that is conjugate linear w.r.t. the first argument and linear w.r.t. the second one. Let $T : \mathcal{H} \supset \mathrm{Dom}(T) \to \mathcal{H}$ be a linear operator defined on a linear subspace[5] $\mathrm{Dom}(T) \subset \mathcal{H}$, dense in $\mathcal{H}$.

Let $\mathrm{Dom}(T^*)$ be a linear subspace of the space $\mathcal{H}$, consisting of all $\varphi \in \mathcal{H}$ such that the map $\psi \to \langle T\psi, \varphi \rangle_{\mathcal{H}}$ is a bounded linear functional on $\mathrm{Dom}(T)$. Since $\overline{\mathrm{Dom}(T)} = \mathcal{H}$, the Riesz lemma [144] implies that for every $\varphi \in \mathrm{Dom}(T^*)$ there exists a unique element $\chi \in \mathcal{H}$ such that

$$\langle T\psi, \varphi \rangle_{\mathcal{H}} = \langle \psi, \chi \rangle_{\mathcal{H}}, \ \forall \psi \in \mathcal{H} \, .$$

Define the linear operator T^* *adjoint* to T by the formula $T^* \varphi = \chi$.

The operator T is called *symmetric* if $T \subset T^*$, i.e., if $\mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$ and $T\varphi = T^* \varphi$, $\forall \varphi \in \mathrm{Dom}(T)$. The operator T is called *self-adjoint* if $T = T^*$; in other words if T is symmetric and $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$.

In many cases it is hard to find a domain of self-adjointness for a symmetric operator. This motivates the notion of *essential self-adjointness*. The linear subspace

$$\Gamma(T) := (\, (\varphi, \psi) \in \mathcal{H} \oplus \mathcal{H} \,|\ \varphi \in \mathrm{Dom}(T),\ \psi = T\varphi)$$

of the space $\mathcal{H} \oplus \mathcal{H}$ is called the *graph* of T. If the closure $\overline{\Gamma(T)}$ of $\Gamma(T)$ w.r.t. the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H} \oplus \mathcal{H}}$ is a graph of some linear operator $\overline{T}$ (evidently it is equivalent to the fact that in $\overline{\Gamma(T)}$ there are no elements of the form $(0, \psi)$, $0 \neq \psi \in \mathcal{H}$) , then T is called *closable* and $\overline{T}$ is the *closure* of T. For every operator T the operator T^* is closed and T is closable iff $\overline{\mathrm{Dom}(T^*)} = \mathcal{H}$. In the latter case $\overline{T} = T^{**}$ (theorem VIII.1 in [144]). For a symmetric operator T it holds $\mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$ and therefore a symmetric operator is closable.

The following theorem is well known, see for example [85]:

Theorem 2.5. *The following conditions for a symmetric operator T are equivalent*

1. $\overline{T}$ *is self-adjoint ;*
2. T *has a unique self-adjoint extension ;*
3. $T^* = T^{**}$*;*
4. T^* *is symmetric .*

If any (and thus all) of above conditions holds, then T is called *essentially self-adjoint*. If $T|_{\mathcal{D}}$ is essentially self-adjoint for some linear subspace $\mathcal{D}$, then this subspace is called an *essential domain* for T.

All these facts are valid for unbounded as well as for bounded operators T. In the latter case $\mathrm{Dom}(T) = \mathcal{H}$. Quantum mechanical Hamiltonians are symmetric and most of them are unbounded. Due to the Hellinger-Toeplitz theorem [144] an everywhere defined symmetric operator $\mathcal{H} \to \mathcal{H}$ is bounded. Therefore, in quantum mechanics the situation $\mathrm{Dom}(T) \neq \mathcal{H}$ is typical.

[5] We do not suppose a linear subspace in $\mathcal{H}$ to be closed.

A *sesquilinear form* in $\mathcal{H}$ is a map

$$q: \ \mathrm{Dom}(q) \times \mathrm{Dom}(q) \to \mathbb{C} ,$$

where $\mathrm{Dom}(q)$ is a dense linear subspace of $\mathcal{H}$, q is conjugate linear w.r.t. the first argument and linear w.r.t. the second one. A sesquilinear form q is symmetric if $q(\varphi,\psi) = \overline{q(\psi,\varphi)}$. A symmetric form q is *positive* if $q(\varphi,\varphi) \geqslant 0$, $\forall\varphi \in \mathrm{Dom}(q)$ and *semibounded* if $q(\varphi,\varphi) \geqslant c\|\varphi\|^2$, $\forall\varphi \in \mathrm{Dom}(q)$, where $c \in \mathbb{R}$ and $\|\cdot\|$ is the norm in $\mathcal{H}$, generated by the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. A symmetric operator T is *semibounded* if $\langle\varphi, T\varphi\rangle \geqslant c\|\varphi\|^2$, $\forall\varphi \in \mathrm{Dom}(T), c \in \mathbb{R}$.

Let q be a semibounded sesquilinear form and $q(\varphi,\varphi) \geqslant c\|\varphi\|^2$. It is *closed* if $\mathrm{Dom}(q)$ is closed w.r.t. the scalar product $\langle\varphi,\psi\rangle_+ := q(\varphi,\psi) + (1-c)\langle\varphi,\psi\rangle_{\mathcal{H}}$. In this case a linear subspace $\mathcal{D} \subset \mathrm{Dom}(q)$ is an *essential domain* of q if the closure of $\mathcal{D}$ w.r.t. $\langle\cdot,\cdot\rangle_+$ equals $\mathrm{Dom}(q)$. If q is not closed one can complete $\mathcal{H}$ w.r.t. the scalar product $\langle\cdot,\cdot\rangle_+$ and obtain the Hilbert space $\mathcal{H}_+$. Clearly the form q is extended up to the bounded sesquilinear form $\hat{q}$ on $\mathcal{H}_+$. Since $\|\varphi\| \leqslant \|\varphi\|_+$ for $\varphi \in \mathrm{Dom}(q)$, the inclusion $\mathrm{Dom}(q) \to \mathcal{H}$ is extended up to the bounded map $j: \mathcal{H}_+ \to \mathcal{H}$ with the norm less or equal 1. If j is injective one can consider $\mathcal{H}_+$ as a linear subspace of $\mathcal{H}$ and the form $\hat{q}$ as the closed form in $\mathcal{H}$ with $\mathrm{Dom}(\hat{q}) = \mathcal{H}_+ \subset \mathcal{H}$. In this case the form q is called *closable* and the form $\hat{q}$ with $\mathrm{Dom}(\hat{q}) = \mathcal{H}_+ \subset \mathcal{H}$ is called the *closure* of q .

For example, the form $q(\varphi,\psi) := \overline{\varphi(0)}\psi(0)$, where $\varphi,\psi \in C_0^\infty(\mathbb{R},\mathbb{C}) \subset \mathcal{H} := \mathcal{L}^2(\mathbb{R}, dx)$, is not closable since for it $\mathcal{H}_+ = ((\psi,a) \,|\, \psi \in \mathcal{L}^2(\mathbb{R},dx), a \in \mathbb{C})$, the corresponding map j acts as $j(\psi,a) = \psi$ and is not injective ([144] Sect. VIII.6, [145] Sect. X.3).

The following correspondence between operators and sesquilinear form in $\mathcal{H}$ is a key instrument for studying the self-adjointness. Let T be a self-adjoint operator in $\mathcal{H}$. Due to the spectral theorem [144] there is an isomorphism

$$\mathcal{H} \cong \bigoplus_{k=1}^{N} \mathcal{L}^2(\mathbb{R}, d\mu_k) \tag{2.23}$$

of Hilbert spaces such that T corresponds to the operator of multiplication by the coordinate $x \in \mathbb{R}$ in all spaces $\mathcal{L}^2(\mathbb{R}, d\mu_k)$. Here $d\mu_k$ are some positive measures on $\mathbb{R}$ and $N \in \mathbb{N} \cup \infty$. The isomorphism (2.23) represents an every element $\psi \in \mathcal{H}$ as the sequence (finite or infinite) $\psi_k(x) \in \mathcal{L}^2(\mathbb{R}, d\mu_k)$, $k = 1,\ldots,N$. Define the domain $\mathrm{Dom}_q(T)$

$$\mathrm{Dom}_q(T) := \left((\psi_k)_{k=1}^N \mid \sum_{k=1}^{N} \int_{-\infty}^{\infty} |x|\, |\psi_k(x)|^2 d\mu_k < \infty \right) \tag{2.24}$$

and the sesquilinear form q_T

$$q_T(\varphi,\psi) := \sum_{k=1}^{N} \int_{-\infty}^{\infty} x\overline{\varphi_k(x)}\psi_k(x) d\mu_k$$

for $\varphi,\psi \in \mathrm{Dom}(q_T) := \mathrm{Dom}_q(T)$.

The form q_T is *generated by the operator* T. Evidently, $\mathrm{Dom}(T) \subset \mathrm{Dom}_q(T)$, but the latter domain could be bigger than the former. Clearly, if T is a semibounded self-adjoint operator, then the form q_T is semibounded. The constructions from proofs of the following two theorems from [144] and [145] will be used below for Schrödinger operators.

Theorem 2.6. *If T is a semibounded self-adjoint operator, then the form q_T is closed and every essential domain of T is an essential domain for q_T. Conversely, an every closed semibounded sesquilinear form q equals q_T for a unique self-adjoint operator T. Moreover if $q(\varphi,\varphi) \geqslant c\|\varphi\|^2$, then $T \geqslant c\,\mathrm{id}$.*

Proof. For scalar operators and corresponding sesquilinear forms, proportional to the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, both claims of the theorem are obvious. Therefore, without loss of generality, one can suppose that $T \geqslant \mathrm{id}$ and $q(\cdot,\cdot) \geqslant \langle\cdot,\cdot\rangle_{\mathcal{H}}$.

Let $T \geqslant \mathrm{id}$ be a self-adjoint operator. Then $\langle\varphi,\psi\rangle_+ := \langle\varphi, T\psi\rangle_{\mathcal{H}}$ is a scalar product on $\mathrm{Dom}(T)$. Denote by $\mathcal{H}_+$ the closure of $\mathrm{Dom}(T)$ w.r.t. $\langle\varphi,\psi\rangle_+$.

Let $\imath : \mathrm{Dom}(T) \to \mathcal{H}$ be the inclusion. Due to $\|\cdot\|_{\mathcal{H}} \leqslant \|\cdot\|_+$, the map $\imath$ can be uniquely extended to the map $\hat{\imath} : \mathcal{H}_+ \to \mathcal{H}$ with a norm less or equal to 1.

Show that $\hat{\imath}$ is injective. Let $\hat{\imath}(\varphi) = 0$, then there exists a sequence $\varphi_i \in \mathrm{Dom}(T)$ such that $\|\varphi - \varphi_i\|_+ \to 0$ and $\|\hat{\imath}(\varphi_i)\|_{\mathcal{H}} = \|\varphi_i\|_{\mathcal{H}} \to 0$, as $i \to \infty$. Therefore, it holds

$$\|\varphi\|_+^2 = \lim_{i,j\to\infty} \langle\varphi_i,\varphi_j\rangle_+ = \lim_{j\to\infty}\lim_{i\to\infty} \langle\varphi_i, T\varphi_j\rangle_{\mathcal{H}} = \lim_{j\to\infty} \langle 0, T\varphi_j\rangle_{\mathcal{H}} = 0$$

that implies $\varphi = 0$. Thus, the map $\hat{\imath}$ is injective and $\mathcal{H}_+ \subset \mathcal{H}$.

If $\psi \in \mathrm{Dom}(T)$, then in notations of (2.24)

$$\sum_{k=1}^{N} \int_{-\infty}^{\infty} |x|\,|\psi_k(x)|^2 d\mu_k = \langle\psi, T\psi\rangle = \langle\psi,\psi\rangle_+ \ .$$

Therefore, $\mathrm{Dom}_q(T)$ is a closure of $\mathrm{Dom}(T)$ in $\mathcal{H}$ w.r.t. the inner product $\langle\cdot,\cdot\rangle_+$. Since $\mathcal{H}_+ \subset \mathcal{H}$, one gets $\mathrm{Dom}_q(T) = \mathcal{H}_+$ and q_T is closed.

Since an every essential domain for T is dense in $\mathrm{Dom}(T)$ w.r.t. the norm $\|\psi\|_T := \|T\psi\| \geqslant \|\psi\|_+$, the first claim of the theorem is proved.

Conversely, let $q(\cdot,\cdot) \geqslant \langle\cdot,\cdot\rangle_{\mathcal{H}}$ be a closed sesquilinear form. Then $\mathcal{H}_+ := \mathrm{Dom}(q) \subset \mathcal{H}$ is a Hilbert space w.r.t. the scalar product $\langle\varphi,\psi\rangle_+ := q(\varphi,\psi)$. Due to the Riesz lemma the identity

$$\langle\varphi, \jmath(\psi)\rangle_+ = \langle\varphi,\psi\rangle_{\mathcal{H}},\ \forall\varphi \in \mathcal{H}_+$$

defines the linear operator $\jmath : \mathcal{H} \to \mathcal{H}_+ \subset \mathcal{H}$ with a norm less or equal to 1.

This operator is injective, otherwise there exists $\psi \in \mathcal{H}_+$, $\psi \neq 0$ such that

$$0 = \langle\varphi, 0\rangle_+ = \langle\varphi, \jmath(\psi)\rangle_+ = \langle\varphi,\psi\rangle_{\mathcal{H}},\ \forall\varphi \in \mathcal{H}_+ \ ,$$

which contradicts to the fact that $\mathcal{H}_+$ is dense in $\mathcal{H}$. Also, the set $\mathrm{Im}\,\jmath$ is dense in $\mathcal{H}_+$ (and therefore in $\mathcal{H}$), otherwise there exists $\varphi \in \mathcal{H}$, $\varphi \neq 0$ such that

$$0 = \langle \varphi, \jmath(\psi) \rangle_+ = \langle \varphi, \psi \rangle_{\mathcal{H}}, \ \forall \psi \in \mathcal{H}_+ \ ,$$

which again contradicts to the fact that $\mathcal{H}_+$ is dense in $\mathcal{H}$.

The identity

$$\begin{aligned} \langle \varphi, \jmath(\psi) \rangle_{\mathcal{H}} &= \overline{\langle \jmath(\psi), \varphi \rangle_{\mathcal{H}}} = \overline{\langle \jmath(\psi), \jmath(\varphi) \rangle_+} = \langle \jmath(\varphi), \jmath(\psi) \rangle_+ \\ &= \langle \jmath(\varphi), \psi \rangle_{\mathcal{H}} \ , \ \forall \varphi, \psi \in \mathcal{H} \end{aligned}$$

shows that the operator $\jmath$ is symmetric and being bounded, it is self-adjoint. Due to the functional calculus of self-adjoint operators and injectivity of $\jmath$, the operator $T := \jmath^{-1}|_{\operatorname{Im} \jmath}$ is self-adjoint with $\operatorname{Dom}(T) = \operatorname{Im} \jmath$.

Also, it holds

$$\langle \psi, \jmath(\psi) \rangle_{\mathcal{H}} = \langle \jmath(\psi), \jmath(\psi) \rangle_+ \geqslant 0$$

and therefore $0 \leqslant \jmath \leqslant \mathrm{id}$. This implies $T \geqslant \mathrm{id}$.

Since

$$q_T(\varphi, \psi) = \langle \varphi, T\psi \rangle_{\mathcal{H}} = \langle \varphi, \jmath(T\psi) \rangle_+ = \langle \varphi, \psi \rangle_+ = q(\varphi, \psi), \ \forall \varphi, \psi \in \operatorname{Dom}(T) \ ,$$

the closed forms q_T and q coincide on the domain $\operatorname{Dom}(T)$, which is dense both in $\operatorname{Dom}(q) = \mathcal{H}_+$ and $\operatorname{Dom}(q_T)$. Therefore, one gets $q_T = q$.

Let now $q_T = q_{T'}$ for another self-adjoint operator T'. This yields

$$\langle \varphi, \jmath(T'\psi) \rangle_+ = \langle \varphi, T'\psi \rangle_{\mathcal{H}} = q_{T'}(\varphi, \psi) = \langle \varphi, \psi \rangle_+ \ , \ \forall \varphi, \psi \in \operatorname{Dom}(T')$$

and therefore

$$\jmath \circ T'|_{\operatorname{Dom}(T')} = \mathrm{id}|_{\operatorname{Dom}(T')} \ .$$

Thus, $\operatorname{Dom}(T') \subset \operatorname{Im} \jmath = \operatorname{Dom}(T)$ and T is a self-adjoint extension of T'. Due to Theorem 2.5 it holds $T = T'$. □

Theorem 2.7. *Let $T \geqslant c\,\mathrm{id}$ be a semibounded symmetric operator and $q(\varphi, \psi) := \langle \varphi, T\psi \rangle_{\mathcal{H}}$ for $\varphi, \psi \in \operatorname{Dom}(H)$. Then q is a closable sesquilinear form and its closure $\bar{q}$ equals q_{T_F} for a unique self-adjoint operator T_F, which is a semibounded extension of T. Also, it holds $T_F \geqslant c\,\mathrm{id}$ and T_F is the unique self-adjoint extension of T with a domain contained in $\operatorname{Dom}(\bar{q})$.*

Proof. Again, without loss of generality, one can consider a symmetric operator $T \geqslant \mathrm{id}$. By the same proof as for the first claim of the previous theorem one gets that $\mathcal{H}_+ \subset \mathcal{H}$, where $\mathcal{H}_+$ is the closure of $\operatorname{Dom}(H)$ w.r.t. the scalar product $\langle \cdot, \cdot \rangle_+ := q(\cdot, \cdot)$. This means that q is closable.

Let $\bar{q}$ be the closure of q. Evidently, $\bar{q}(\varphi, \varphi) \geqslant c\|\varphi\|^2$. Due to the previous theorem there exists a unique self-adjoint operator T_F such that $\bar{q} = q_{T_F}$ and

$$\bar{q}(\varphi, \psi) = \langle \varphi, T_F\psi \rangle_{\mathcal{H}}, \ \forall \varphi, \psi \in \operatorname{Dom}(T_F) \subset \operatorname{Dom}(\bar{q}) \ .$$

Also, this operator obeys the inequality $T_F \geqslant c\,\mathrm{id}$. Since $\bar{q}$ is continuous in $\mathcal{H}_+$, it holds

$$\langle T\varphi, \psi \rangle_{\mathcal{H}} = \bar{q}(\varphi, \psi) = \langle \varphi, T_F\psi \rangle_{\mathcal{H}}, \ \text{for } \varphi \in \operatorname{Dom}(T), \ \forall \psi \in \operatorname{Dom}(T_F) \ .$$

Thus, one gets $\varphi \in \mathrm{Dom}(T_F^*) = \mathrm{Dom}(T_F)$, $T\varphi = T_F^*\varphi = T_F\varphi$ and the operator T_F is an extension of T.

Now let $\widetilde{T}$ be a symmetric extension of T such that $\mathrm{Dom}(\widetilde{T}) \subset \mathrm{Dom}(\bar{q})$. For $\psi \in \mathrm{Dom}(\widetilde{T})$, $\varphi \in \mathrm{Dom}(T)$ it holds

$$\bar{q}(\varphi, \psi) = \langle T\varphi, \psi\rangle_{\mathcal{H}} = \langle \widetilde{T}\varphi, \psi\rangle_{\mathcal{H}} = \langle \varphi, \widetilde{T}\psi\rangle_{\mathcal{H}} .$$

Since $\bar{q}$ is continuous in $\mathcal{H}_+$, one gets

$$\bar{q}(\varphi, \psi) = \langle \varphi, \widetilde{T}\psi\rangle_{\mathcal{H}} =: \tilde{q}(\varphi, \psi), \ \forall \psi \in \mathrm{Dom}(\widetilde{T}) .$$

Therefore, $\bar{q}$ is a closure of $\tilde{q}$ and the proof above, applied to the operator $\widetilde{T}$ (instead of T) gives $\widetilde{T} \subset T_F$.

If additionally $\widetilde{T}$ is self-adjoint, then Theorem 2.5 implies $\widetilde{T} = T$, which completes the proof. □

The operator T_F from the latter theorem is called the *Friedrichs extension* of T.

The general method for proving the self-adjointness or essential self-adjointness of some symmetric operator is a representation of this operator as a "perturbation" of a self-adjoint operator. Different meanings of the term "perturbation" lead to different results. The most well-known examples are below.

Theorem 2.8 (Kato-Rellich; [145], theorem X.12). *Let A be a self-adjoint and B be a symmetric operator such that*

1. $\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ *;*
2. *there are real constants $a < 1$ and $b > 0$ such that*

$$\|B\psi\| \leqslant a\|A\psi\| + b\|\psi\|, \ \forall \psi \in \mathrm{Dom}(A) .$$

Then $A + B$ defined on $\mathrm{Dom}(A)$ is self-adjoint and essentially self-adjoint on every essential domain of A.

Theorem 2.9 (KLMN; [145], theorem X.17). *Let A be a self-adjoint positive operator and q be a symmetric sesquilinear form on $\mathrm{Dom}_q(A)$ such that*

$$|q(\varphi, \varphi)| \leqslant a\langle \varphi, A\varphi\rangle + b\langle \varphi, \varphi\rangle, \ \forall \varphi \in \mathrm{Dom}(A)$$

for real constants $a < 1$ and $b > 0$. Then there exists a unique self-adjoint operator B with $\mathrm{Dom}_q(B) = \mathrm{Dom}_q(A)$ such that

$$\langle \varphi, B\varphi\rangle = \langle \varphi, A\varphi\rangle + q(\varphi, \varphi), \ \forall \varphi \in \mathrm{Dom}_q(B) .$$

Any essential domain of A is an essential domain for B.

2.3.2 Self-Adjointness of Schrödinger Operators on Riemannian Spaces

Let now M^n be a complete C^∞ Riemannian manifold with a metric g and μ be the measure on M^n, induced by g. This means that locally one has

$$d\mu = \sqrt{\det g_{ij}} dx^1 \cdot \ldots \cdot dx^n, \tag{2.25}$$

if locally $g = g_{ij} dx^i dx^j$. Let $\triangle = \triangle_g$ be the Laplace–Beltrami operator on M^n.

The most natural "unperturbed" part of a Schrödinger operator

$$-\triangle + V \tag{2.26}$$

with a real scalar potential V is the operator $-\triangle$. It is essentially self-adjoint with $\mathrm{Dom}(-\triangle) = C_c^\infty(M^n, \mathbb{C})$ ([188], theorem 2.4). Note that the operator $\triangle$ naturally acts not only on functions, but also on differential forms and tensor fields.

In Euclidean case the approach to a proof of the self-adjointness of the operator (2.26) as the perturbation of $-\triangle$ was demonstrated in [145] on different physical examples (see also the review in [175]). The review of different results concerning the self-adjointness of Schrödinger operators on Riemannian manifolds can be found in [30]. Here we formulate the simplified result from [43], sufficient for most concrete spaces and potentials, considered below.

Denote by H_0 the operator (2.26) defined on the space $C_c^\infty(M^n, \mathbb{C})$ of smooth functions with a compact support. This operator is evidently symmetric in the Hilbert space $\mathcal{L}^2(M^n, d\mu)$ and therefore closable. Let $W_{\mathrm{loc}}^{k,l}(M^n, d\mu)$ be a linear subspace of the space $\mathcal{L}^2(M^n, d\mu)$ consisting of complex-valued functions on M^n, whose partial derivatives (in the sense of distributions) up to the order k are in $\mathcal{L}^2_{\mathrm{loc}}(M^n, d\mu)$.

Theorem 2.10 ([43]). *Let $V \in \mathcal{L}^p_{\mathrm{loc}}(M^n, d\mu)$, $n \geqslant 2$, where $p = 2$ for $n = 2, 3$, $p > n/2$ for $n \geqslant 4$ and $V \geqslant -c$ outside of a compact set, for some constant $c > 0$. Then*

1. *the operator H_0 is essentially self-adjoint;*
2. *the operator $\overline{H}_0$ is self-adjoint with*

$$\mathrm{Dom}(\overline{H}_0) = \Big(u \in W^{2,2}_{\mathrm{loc}}(M^n, d\mu) \cap \mathcal{L}^2(M^n, d\mu) \Big| \; (-\triangle u + Vu)_{\mathrm{dist}} \in \mathcal{L}^2(M^n, d\mu)\Big)\,, \tag{2.27}$$

where subscript "dist" means that a differential operator is understood in the sense of distributions,

3. *for an element $u \in \mathrm{Dom}(\overline{H}_0)$ the following inclusions are valid:*

$$|\nabla u| \in \mathcal{L}^2(M^n, d\mu), \; |V|^{1/2} u \in \mathcal{L}^2(M^n, d\mu)\,.$$

Conditions on p were weaken in [121] for some special type of Riemannian manifolds. The *exponential geodesic map* $\exp_x : T_x M \mapsto M$ is defined by the formula $\exp_x X = \gamma(1)$, where $\gamma(s)$ is the geodesic, starting at the point $x \in M$ with the initial speed $\gamma'(0) = X \in T_xM$ and s being its arc length. The exponential geodesic map is a diffeomorphism of a ball in T_xM of a radius r with the center 0 onto a neighborhood $U_{x,r}$ of x in M. Let r_x be the supremum of possible radii (may be ∞) of such balls in T_xM and $r_{inj} := \inf_{x \in M} r_x$.

Definition 2.1 ([35, 173]). *A Riemannian manifold M is called a manifold of a bounded geometry if the following conditions are satisfied*

1. $r_{inj} > 0$,
2. $|\nabla^k R| \leqslant C_k$, $k = 0, 1, 2, \dots$,

where ∇ is the Levi-Civita connection, R is the curvature tensor, $|\cdot|$ is the norm in the space of tensors, generated by the Riemannian metric in T_xM, and C_k are real constants.

Note that the first condition implies the completeness of M. Evidently, every homogeneous or compact Riemannian manifold is a manifold of a bounded geometry. The following theorem is a result from [121], restricted for the scalar case.

Theorem 2.11. *Let M^n be a manifold of a bounded geometry and the potential V can be represented in the form $V = V_1 + V_2$, where real-valued functions V_1, V_2 are as follows: $0 \leqslant V_1 \in \mathcal{L}^1_{\text{loc}}(M^n, d\mu)$, $0 \geqslant V_2 \in \mathcal{L}^p(M^n, d\mu)$ for $p = n/2$ if $n \geqslant 3$, for $p > 1$ if $n = 2$ and for $p = 1$ if $n = 1$.*

Then the operator $H = -\triangle + V$ is self-adjoint with the domain:

$$\text{Dom}(H) = \left(u \in W^{1,2}(M^n, d\mu) \middle| \int_M V_1 |u|^2 d\mu < +\infty,\ Hu \in \mathcal{L}^2(M^n, d\mu) \right), \tag{2.28}$$

where Hu is understood in the sense of distributions. Also, it holds $Vu \in \mathcal{L}^1_{\text{loc}}(M^n, d\mu)$ for $u \in \text{Dom}(H)$.

If the potential V is not in $\mathcal{L}^1_{\text{loc}}(M^n, d\mu)$, then theorems 2.10 and 2.11 are not applicable. If instead V is bounded from below, one can try to restrict the Schrödinger operator onto some submanifold M' of M^n such that $V|_{M'} \in \mathcal{L}^1_{\text{loc}}(M', d\mu)$ and construct the Friedrichs self-adjoin extension of $-\triangle + V$ from the initial domain $C_c^\infty(M', \mathbb{C})$. This procedure is physically motivated for instance in the case when $V \to +\infty$ near the boundary of M' and therefore wave functions should vanish near this boundary.

Let us turn to the accurate mathematical description. Let M' be an open connected submanifold of M^n of dimension n. We do not suppose that M' is complete w.r.t. the Riemannian structure induced by the Riemannian structure on M^n. Let $V \geqslant C \in \mathbb{R}$ be a real-valued function from $\mathcal{L}^1_{\text{loc}}(M', d\mu)$ and $H_0 = -\triangle + V$ be a Schrödinger operator with the domain $C_c^\infty(M', \mathbb{C})$. Without loss of generality we suppose that $C = 1$. Let $H_F \geqslant \text{id}$ be the Friedrichs extension of H_0, constructed in Theorem 2.7 in the abstract case. Now one needs a precise description of $\text{Dom}(H_F)$.

Denote by $\operatorname{grad} f$ the gradient of a function f on M^n, defined in local coordinates by the expression

$$g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i} \ .$$

Evidently, the closure of the sesquilinear form q_{H_0}, associated with the operator H_0, is

$$q_{H_F}(\varphi, \psi) = \int_{M'} (g(\operatorname{grad} \bar{\varphi}, \operatorname{grad} \psi) + V \bar{\varphi} \psi) \, d\mu \tag{2.29}$$

with $\operatorname{Dom}(q_{H_F}) \subset \mathcal{L}^2(M', d\mu)$ being a closure of $C_c^\infty(M', \mathbb{C})$ w.r.t. the inner product (2.29).

According to the construction in the proof of Theorem 2.6 the operator H_F is defined by the identity

$$\int_{M'} (g(\operatorname{grad} \bar{\varphi}, \operatorname{grad} \psi) + V \bar{\varphi} \psi) \, d\mu$$
$$= \int_{M'} \bar{\varphi} H_F \psi d\mu, \ \forall \varphi \in \operatorname{Dom}(q_{H_F}), \psi \in \operatorname{Dom}(H_F) \ .$$

Thus, one gets

$$H_F \psi = (- \triangle \psi + V\psi)_{\text{dist}} \ , \ \psi \in \operatorname{Dom}(H_F) \ . \tag{2.30}$$

Let $- \triangle_F + \text{id}$ be the Friedrichs extension (evidently nonnegative) of the operator

$$(- \triangle + \text{id})|_{C_c^\infty(M', \mathbb{C})}$$

and $\operatorname{Dom}(q_{-\triangle_F + \text{id}})$ be the domain of associated sesquilinear form. It is the closure of $C_c^\infty(M', \mathbb{C})$ w.r.t. the inner product

$$q_{-\triangle_F + \text{id}}(\varphi, \psi) = \int_{M'} (g(\operatorname{grad} \bar{\varphi}, \operatorname{grad} \psi) + \bar{\varphi} \psi) \, d\mu \ . \tag{2.31}$$

Let also $\operatorname{Dom}(q_V)$ be the closure of $C_c^\infty(M', \mathbb{C})$ w.r.t. the inner product

$$q_V(\varphi, \psi) = \int_{M'} \bar{\varphi} V \psi d\mu \ .$$

The following theorem is the direct generalization of theorem X.32 from [145] for Riemannian manifolds. For $M' = \mathbf{H}^n(\mathbb{R})$ it was announced in [159].

Theorem 2.12. *The domain of the operator H_F is*

$$\left(\psi \in \operatorname{Dom}(q_{-\triangle_F + \text{id}}) \mid V\psi \in \mathcal{L}^1_{\text{loc}}(M', d\mu); (- \triangle \psi + V\psi)_{\text{dist}} \in \mathcal{L}^2(M', d\mu)\right) \tag{2.32}$$

and H_F acts by formula (2.30)*.*

For the proof of this theorem one needs some preliminary propositions. The first proposition is the generalization of the *Kato inequality* for Riemannian spaces (see theorem 5.7 from [30]), restricted onto the scalar case (cf. theorem X.27 from [145]).

Proposition 2.6. *Let* $u \in \mathcal{L}^1_{\text{loc}}(M', d\mu)$ *and* $(\triangle u)_{\text{dist}} \in \mathcal{L}^1_{\text{loc}}(M', d\mu)$*, then it holds*

$$(\triangle |u|)_{\text{dist}} \geqslant \operatorname{Re}\left((\operatorname{sign} u)(\triangle u)_{\text{dist}}\right) , \tag{2.33}$$

where

$$(\operatorname{sign} u)(\mathbf{x}) := \begin{cases} \bar{u}(\mathbf{x})/|u(\mathbf{x})|, & \text{if} \quad u(\mathbf{x}) \neq 0 \\ 0, & \text{otherwise} . \end{cases}$$

Here the inequality (2.33) *is understood in the sense of distributions.*

The second preliminary proposition is the consequence of theorem 2.2 from [122]. Let a set $\mathcal{L}^2(M', d\mu)^+$ consists of all real-valued functions from $\mathcal{L}^2(M', d\mu)$ that are positive almost everywhere w.r.t. the measure μ. Similarly $C_c^\infty(M')^+$ denotes the set of all nonnegative functions from $C_c^\infty(M')$.

Proposition 2.7. *For any function* u *from* $\operatorname{Dom}(q_{-\triangle_F + \text{id}}) \cap \mathcal{L}^2(M', d\mu)^+$ *there exists a sequence* $u_k \in C_c^\infty(M')^+$ *such that* $\|u_k - u\|_{-\triangle_F + \text{id}} \to 0$ *as* $k \to \infty$*, where* $\|\cdot\|_{-\triangle_F + \text{id}}$ *is the norm, associated with the positive sesquilinear form* (2.31).

The third preliminary proposition is the particular case of Lemma 2.12 from [122].

Proposition 2.8. *The bounded operator* $(-\triangle_F + \text{id})^{-1}$ *maps the set* $\mathcal{L}^2(M', d\mu)^+$ *into itself.*

Now one can give a proof of Theorem 2.12. Until inequality (2.36) the following proof repeats *mutatis mutandis* the proof of theorem X.32 from [145]. The last part of the proof follows ideas from [122].

Proof of theorem 2.12. Let $\widetilde{H}$ be an operator with $\operatorname{Dom}(\widetilde{H})$ given by (2.32) and

$$\widetilde{H}\psi := (-\triangle \psi + V\psi)_{\text{dist}} , \; \psi \in \operatorname{Dom}(\widetilde{H}) .$$

Firstly, we will show that $\widetilde{H}$ is an extension of H_F. Since $-\triangle_F$ and V are nonnegative operators, it holds

$$\operatorname{Dom}(H_F) \subset \operatorname{Dom}(q_{H_F}) = \operatorname{Dom}(q_{-\triangle_F + \text{id}}) \cap \operatorname{Dom}(q_V).$$

Let $\varphi \in \operatorname{Dom}(H_F)$, then $V^{1/2}\varphi \in \mathcal{L}^2(M', d\mu)$. Since also $V^{1/2} \in \mathcal{L}^2_{\text{loc}}(M', d\mu)$, the Cauchy inequality implies

$$V\varphi = V^{1/2}(V^{1/2}\varphi) \in \mathcal{L}^1_{\text{loc}}(M', d\mu) . \tag{2.34}$$

Thus, due to (2.30) one gets $\varphi \in \operatorname{Dom}(\widetilde{H})$, $\operatorname{Dom}(H_F) \subset \operatorname{Dom}(\widetilde{H})$ and

$$\widetilde{H}\Big|_{\operatorname{Dom}(H_F)} = H_F . \tag{2.35}$$

Conversely, let $\varphi \in \operatorname{Dom}(\widetilde{H})$. Since H_F is a self-adjoint operator and $H_F \geqslant \text{id}$, there is a function $\psi \in \operatorname{Dom}(H_F)$ such that $\widetilde{H}\varphi = H_F\psi$. Due to

(2.35) it holds $\widetilde{H}\eta = 0$, where $\eta := \varphi - \psi \in \mathrm{Dom}(\widetilde{H}) \subset \mathrm{Dom}(q_{-\triangle_F + \mathrm{id}})$. Due to (2.34) this means that

$$(-\triangle\,\eta)_{\mathrm{dist}} = \widetilde{H}\eta - V\eta = -V\eta \in \mathcal{L}^1_{\mathrm{loc}}(M', d\mu)$$

and Proposition 2.6 implies

$$(\triangle|\eta|)_{\mathrm{dist}} \geqslant \mathrm{Re}\,((\mathrm{sign}\,\eta)V\eta) = V|\eta| \geqslant |\eta|\ . \tag{2.36}$$

This yields

$$\langle(-\triangle +1)w, |\eta|\rangle_{\mathcal{L}^2(M',d\mu)} \leqslant 0,\ \forall w \in C_c^\infty(M')^+\ . \tag{2.37}$$

Due to Proposition 2.8 the function $f := (-\triangle_F + \mathrm{id})^{-1}|\eta|$ belongs to $\mathcal{L}^2(M', d\mu)^+ \cap \mathrm{Dom}(-\triangle_F)$. Proposition 2.7 implies therefore the existence of a sequence $f_k \in C_c^\infty(M')^+$ such that

$$\begin{aligned}\lim_{k\to\infty} q_{-\triangle_F+\mathrm{id}}(f_k, |\eta|) &= q_{-\triangle_F+\mathrm{id}}(f, |\eta|)\\ &= q_{-\triangle_F+\mathrm{id}}((-\triangle_F+\mathrm{id})^{-1}|\eta|, |\eta|) = \|\eta\|^2_{\mathcal{L}^2(M',d\mu)}\ .\end{aligned}$$

From (2.37) one gets $q_{-\triangle_F+\mathrm{id}}(f_k, |\eta|) \leqslant 0$ that gives $\|\eta\|^2_{\mathcal{L}^2(M',d\mu)} \leqslant 0$, i.e., $\eta = 0$. Thus, it holds $\varphi = \psi$ and $\mathrm{Dom}(\widetilde{H}) \subset \mathrm{Dom}(H_F)$ that due to (2.35) completes the proof. □

2.4 General Scheme of Quantum-Mechanical Reduction

Let M be a Riemannian manifold with an action of an isometry group G (not necessarily full) on it. We assume that G-orbits in M of a maximal dimension ℓ are isomorphic to each other, their union is an open dense submanifold M', the set $M\backslash M'$ has zero measure, and it holds $M' = W \times \mathcal{O}$, where $\mathcal{O}$ is a G-orbit of a maximal dimension and W is a submanifold of M' transversal to all G-orbits of the dimension ℓ. This situation is a typical one [31] and we have the isomorphism of measurable sets $(M, \mu) \cong (W, \nu) \times (\mathcal{O}, \mu_G)$, where μ is the G-invariant measure on the manifold M, generated by its metric, ν is some measure on W, and μ_G is a G-invariant measure on $\mathcal{O}$. This implies the isomorphisms of functional spaces (see for example theorem II.10 in [144]):

$$\mathcal{H} := \mathcal{L}^2(M, d\mu) = \mathcal{L}^2(W \times \mathcal{O}, d\nu \otimes d\mu_G) = \mathcal{L}^2(W, d\nu) \otimes \mathcal{L}^2(\mathcal{O}, d\mu_G)\ . \tag{2.38}$$

Under these assumptions, a G-invariant differential operator D on M' admits an explicitly invariant decomposition of the form:

$$D = D_T + \sum_{(i)} D_{(i)} \circ \widetilde{X}_{i_1} \circ \cdots \circ \widetilde{X}_{i_r} \equiv D_T + \sum_{(i)} D_{(i)} \circ \Box_{(i)}\ , \tag{2.39}$$

where as above $\widetilde{X}_i$ is a differential operator of the first order, corresponding to the action of the one parametric subgroup $\exp(tX_i)$, $X_i \in \mathfrak{g}$, of the group

G on the space M'; D_T and $D_{(i)}$ are transversal operators with respect to the submanifold W; here D_T is called the *transversal part* of D (see theorem 3.4 from chapter II, [66]). Operators $\Box_{(i)}$ are invariant ones on G-orbits in the space M'. According to Sect. 2.1 such operators can be naturally expressed in terms of the Lie algebra $\mathfrak{g}$ of the group G. Throughout the book we assume M and G to be connected. Expression (2.39) of invariant differential operators corresponds to a general approach to invariant differential geometrical objects on homogeneous spaces. These objects have the simplest form in the basis of Killing vector fields [17, 92]. For invariant metrics this approach was developed in [7, 9] for applications to infinite-dimensional groups. Note that the representation (2.39) depends on a choice of a submanifold W, transversal to G-orbits.

The group G naturally acts on the space $\mathcal{L}^2(\mathcal{O}, d\mu_G)$ by left-shifts. Let

$$\mathcal{L}^2(\mathcal{O}, d\mu_G) = \oplus_j \mathcal{H}'_j$$

be an expansion of the space $\mathcal{L}^2(\mathcal{O}, d\mu_G)$ such that $\mathcal{H}'_j$ is the sum of all irreducible G-representations from $\mathcal{L}^2(\mathcal{O}, d\mu_G)$ of a fixed irreducible type. Obviously, different G-representations $\mathcal{H}'_j$ have no isomorphic irreducible summands. Since operators $\Box_{(i)}$ commute with G-shifts, they can not transpose spaces $\mathcal{H}'_j$ due to the Schur lemma [88, 212]. Thus, one gets the expansion

$$\mathcal{H} = \mathcal{L}^2(W, d\nu) \otimes \left(\oplus_j \mathcal{H}'_j\right) = \oplus_j \left(\mathcal{L}^2(W, d\nu) \otimes \mathcal{H}'_j\right) \ .$$

of the space $\mathcal{H}$ into the sum of D-invariant subspaces $\mathcal{H}_j := \mathcal{L}^2(W, d\nu) \otimes \mathcal{H}'_j$.[6] Note that for a compact group G spaces $\mathcal{H}'_j$ are finite-dimensional.

If the operator $D = H$ is a Hamiltonian of some quantum mechanical system on a Riemannian manifold M then this construction gives the reduction of a G-invariant quantum mechanical system to a set of its subsystems. This method was described in papers [100, 190, 210] without mentioning the expansion (2.39). On the other hand, it seems to be difficult to manage without the expansion (2.39) while reducing the quantum mechanical system, because representation theory for the group G gives only the formulae for the action of operators $\Box_{(i)}$ in the space $\mathcal{H}'_i$. Without (2.39), the calculation of the action of H in spaces $\mathcal{H}_j$ requires cumbersome computations.

At the same time, expansion (2.39) gives some information on the complexity of reduced subsystems even in the absence of the detailed information about irreducible representations of the group G in the space $\mathcal{L}^2(\mathcal{O}, d\mu_G)$. For example, if all operators $\Box_{(i)}$ in (2.39) commute, they have only common eigenfunctions and the spectral problem for the Hamiltonian H is reduced to a set of spectral problems for some scalar differential operators on the manifold W.

The simplest example for this construction is a one-body Hamiltonian with a central potential in Euclidean space $\mathbf{E}^n$. Here $W = \mathbb{R}_+, \mathcal{O} = \mathbf{S}^{n-1}$ and expansion (2.39) has the form

[6] One should keep in mind that domains of D in these subspaces are some dense subsets.

$$D = H_e = -\frac{1}{2m\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}\right) - \frac{1}{2m\rho^2}\triangle_s + V(\rho) \ ,$$

where ρ is the distance from the center of a potential and $\triangle_s$ is the Laplace-Beltrami operator on $\mathbf{S}^{n-1}$ (everywhere we put $\hbar = 1$). The situation is similar for a one-body Hamiltonian with a central potential in constant curvature spaces. It will be studied in Sect. 6.3.

Consider a more difficult two-body problem on a Riemannian space $\widetilde{M}$ with a Hamiltonian

$$H = -\frac{1}{2m_1}\triangle_1 - \frac{1}{2m_2}\triangle_2 + V =: H_0 + V \tag{2.40}$$

Here H_0 is the free two-particle Hamiltonian, m_1, m_2 are particle masses, $\triangle_i$, $i = 1, 2$ is the Laplace-Beltrami operator on the ith factor of the configuration space $M = \widetilde{M} \times \widetilde{M}$ for this system and V is an interaction potential depending only on a distance between particles. Let G be the identity component of an isometry group for the space $\widetilde{M}$. The group G acts naturally on the space $\widetilde{M} \times \widetilde{M}$ as

$$g: \ (q_1, q_2) \to (gq_1, gq_2), \ g \in G, \ (q_1, q_2) \in Q \times Q \ .$$

The codimension of G-orbits in M in this case is one or greater, since the group G conserves a distance between two points of the space $\widetilde{M}$. In other words, it holds $\dim W \geqslant 1$ for a submanifold $W \subset M$, transversal to G-orbits in M. In Chap. 5 we shall consider the case of two-point homogeneous Riemannian spaces $\widetilde{M} = Q$ for which $\dim W = 1$ and find for H the corresponding expansion (2.39). The main difference of this expansion from the one-particle case is the noncommutativity of an algebra generated by operators $\square_{(i)}$.

3

Algebras of Invariant Differential Operators on Unit Sphere Bundles Over Two-Point Homogeneous Riemannian Spaces

In this chapter we study the algebra $\mathrm{Diff}_I(Q_{\mathbf{S}})$ of invariant differential operators on the unit sphere bundle $Q_{\mathbf{S}}$ over a two-point homogeneous Riemannian space Q. Namely we construct a system of generators and relations for these algebras. These generators will appear in Chap. 5 in explicitly invariant expressions of two-body Hamiltonian operators H on the space Q. On the other hand, the center of the algebra $\mathrm{Diff}_I(Q_{\mathbf{S}})$ commutes with H, i.e., it consists of integrals for the quantum two-body problem on Q.

In Sect. 3.1 we derive some generators and relations for the algebra $\mathrm{Diff}_I(Q_{\mathbf{S}})$ corresponding to an arbitrary two-point homogeneous Riemannian space Q using general information from Sect. 1.2. However, this information is insufficient for deriving all such generators and relations. The further study of generators and relations for $\mathrm{Diff}_I(Q_{\mathbf{S}})$ is based upon models of compact two-point homogeneous Riemannian spaces, described in Sect. 1.3. The description of the algebra $\mathrm{Diff}_I(Q_{\mathbf{S}})$ for noncompact two-point homogeneous Riemannian spaces is obtained using the transformation from Proposition 1.5.

We calculate also some central elements of algebras $\mathrm{Diff}_I(Q_{\mathbf{S}})$, which are used in Chap. 7. The result of Sect. 3.6 will not be used later, but it seems to be interesting from the differential geometry point of view. This chapter is based upon the authors paper [168], but also contains some additional information concerning the centers of algebras $\mathrm{Diff}_I(Q_{\mathbf{S}})$ and the complete form of relation (3.13).

3.1 Invariant Differential Operators on $Q_{\mathbf{S}}$

For an arbitrary Riemannian space M denote by $M_{\mathbf{S}}$ the bundle of unit spheres (in tangent spaces T_xM, $x \in M$) over M. Here we shall specify the construction from Sect. 2.1 for the space $Q_{\mathbf{S}}$, where Q is a two-point homogeneous compact Riemannian space, using Proposition 1.2. In this section we use notations from Sects. 1.2 and 2.1.

Let G be the identity component of the isometry group for Q and K be its stationary subgroup, corresponding to the point $x_0 \in Q$. The group G naturally acts on the space $Q_{\mathbf{S}}$ and this action is transitive due to Theorem 1.1.

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In particular K acts transitively on the unit sphere $\mathbf{S}_{x_0} \subset T_{x_0}Q$. Identify the space $\mathfrak{p}$ from Proposition 1.2 with the space $T_{x_0}Q$. After this identification the action of K on $T_{x_0}Q$ becomes the adjoint action Ad_K on $\mathfrak{p}$. Let K_0 be the subgroup of K, corresponding to the subalgebra $\mathfrak{k}_0 \subset \mathfrak{k}$. Due to relations (1.3) and (1.6) the group K_0 is the stationary subgroup of the group G, corresponding to the point $y := (x_0, \Lambda') \in Q_{\mathbf{S}}$, where $\Lambda' := \dfrac{1}{R}\Lambda$. Using models of compact two-point homogeneous Riemannian spaces we shall see below that the group K_0 is connected.

Let $\tilde{\mathfrak{p}} := \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$. Due to $[\mathfrak{k}_0, \tilde{\mathfrak{p}}] \subset \tilde{\mathfrak{p}}$ the expansion $\mathfrak{g} = \tilde{\mathfrak{p}} \oplus \mathfrak{k}_0$ is reductive. One has $T_y Q_{\mathbf{S}} = T_{x_0}Q \oplus T_{\Lambda'}\mathbf{S}_{x_0}$. Due to Proposition 1.2 one gets $\Lambda' \perp (\mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda})$ and $[f_{\lambda,i}, \Lambda'] = -(2R)^{-1} e_{\lambda,i}$, $i = 1, \ldots, q_1$, $[f_{2\lambda,j}, \Lambda'] = -R^{-1} e_{2\lambda,j}$, $j = 1, \ldots, q_2$. Therefore, the space $\mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$ is identified through the K-action on $T_{x_0}Q$ with the space $T_{\Lambda'}\mathbf{S}_{x_0}$ and the K_0-action on the space $T_y Q_{\mathbf{S}} \simeq \tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$ is again adjoint.

From Proposition 1.2 we see that Ad_{K_0} conserves all summands in the last expansion. On the other hand, the K_0-action on $T_{\Lambda'}\mathbf{S}_{x_0}$ is the differential of K_0-action on $(\Lambda')^\perp \subset T_{x_0}Q$. Since the last action is linear, one sees that the Ad_{K_0}-action in $\mathfrak{p}_\lambda$ is equivalent to it's action in $\mathfrak{k}_\lambda$ and the Ad_{K_0}-action in $\mathfrak{p}_{2\lambda}$ is equivalent to it's action in $\mathfrak{k}_{2\lambda}$. Let $\chi_\lambda : \mathfrak{k}_\lambda \to \mathfrak{p}_\lambda$, $\chi_{2\lambda} : \mathfrak{k}_{2\lambda} \to \mathfrak{p}_{2\lambda}$ be isomorphisms of linear spaces such that $\mathrm{Ad}_{K_0}|_{\mathfrak{p}_\lambda} \circ \chi_\lambda = \chi_\lambda \circ \mathrm{Ad}_{K_0}|_{\mathfrak{k}_\lambda}$ and $\mathrm{Ad}_{K_0}|_{\mathfrak{p}_{2\lambda}} \circ \chi_{2\lambda} = \chi_{2\lambda} \circ \mathrm{Ad}_{K_0}|_{\mathfrak{k}_{2\lambda}}$.

After the substitution $\mathfrak{p} \to \tilde{\mathfrak{p}}$, $\mathfrak{k} \to \mathfrak{k}_0$ one can apply the construction from Sect. 2.1 for calculating generators and relations for the algebra

$$U(\mathfrak{g})^{K_0}/(U(\mathfrak{g})\mathfrak{k}_0)^{K_0} \cong \mathrm{Diff}_G(Q_{\mathbf{S}}) .$$

Let $g_i \in S(\tilde{\mathfrak{p}})$ be independent invariant elements of the Ad_{K_0}-action in $S(\tilde{\mathfrak{p}})$ which generate $S(\tilde{\mathfrak{p}})^{K_0}$. Then elements $\eta \circ \lambda^*(g_i)$ generate the algebra $\mathrm{Diff}_G(Q_{\mathbf{S}})$.

From now in the present chapter we adopt for brevity the following:

Convension 3.1. *Identify isomorphic algebras* $U(\mathfrak{g})^{K_0}/(U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$ *and* $\mathrm{Diff}_G(Q_{\mathbf{S}})$. *Consider elements* $\lambda(g_i) \in U(\mathfrak{g})^{K_0}$ *modulo* $(U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$ *as elements of* $\mathrm{Diff}_G(Q_{\mathbf{S}})$. *Instead of* $g \equiv g' \mod (U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$ *sometimes we shall simply write* $g = g'$. *Since the group* G *is uniquely defined by* Q_S, *we shall write* $\mathrm{Diff}_I(Q_S)$ *instead of* $\mathrm{Diff}_G(Q_S)$ *for the concrete* Q_S.

The element $\Lambda \in \mathfrak{a}$ is invariant w.r.t. the Ad_{K_0}-action in $S(\tilde{\mathfrak{p}})$, since $[\mathfrak{k}_0, \mathfrak{a}] = 0$ and K_0 is connected. Also, the Ad_{K_0}-action is orthogonal w.r.t. the Killing form, which is proportional to the scalar product $\langle\cdot,\cdot\rangle$, so the Ad_{K_0}-action conserves the restrictions of $\langle\cdot,\cdot\rangle$ onto spaces $\mathfrak{p}_\lambda$, $\mathfrak{p}_{2\lambda}$, $\mathfrak{k}_\lambda$, $\mathfrak{k}_{2\lambda}$. Similarly, the Ad_{K_0}-action conserves functions $\langle \chi_\lambda X_1, Y_1\rangle$, $X_1 \in \mathfrak{k}_\lambda$, $Y_1 \in \mathfrak{p}_\lambda$ and $\langle \chi_{2\lambda} X_2, Y_2\rangle$, $X_2 \in \mathfrak{k}_{2\lambda}$, $Y_2 \in \mathfrak{p}_{2\lambda}$. All these functions are invariant elements of $\mathrm{Ad}^*_{K_0}$-action in $S(\mathfrak{p}^*_\lambda \oplus \mathfrak{p}^*_{2\lambda} \oplus \mathfrak{k}^*_\lambda \oplus \mathfrak{k}^*_{2\lambda})$.

Let define linear maps $\chi_\lambda : \mathfrak{k}_\lambda \mapsto \mathfrak{p}_\lambda$ and $\chi_{2\lambda} : \mathfrak{k}_{2\lambda} \mapsto \mathfrak{p}_{2\lambda}$ by the following formulas

$$\chi_\lambda X = 2[\Lambda, X],\ X \in \mathfrak{k}_\lambda,\ \chi_{2\lambda} Y = [\Lambda, Y],\ Y \in \mathfrak{k}_{2\lambda} . \tag{3.1}$$

Indeed, from (3.1) and the identity $\mathrm{Ad}_k \Lambda = \Lambda$, $\forall k \in K_0$ one gets: $\mathrm{Ad}_k \circ \chi_\lambda X = 2[\Lambda, \mathrm{Ad}_k X] = \chi_\lambda \circ \mathrm{Ad}_k X$, $X \in \mathfrak{k}_\lambda$ and $\mathrm{Ad}_k \circ \chi_{2\lambda} Y = [\Lambda, \mathrm{Ad}_k Y] = \chi_{2\lambda} \circ$

$\mathrm{Ad}_k Y$, $Y \in \mathfrak{k}_{2\lambda}$. It is clear that $\chi_\lambda f_{\lambda,i} = e_{\lambda,i}$, $i = 1, \dots, q_1$ and $\chi_{2\lambda} f_{2\lambda,j} = e_{2\lambda,j}$, $j = 1, \dots, q_2$.

Proposition 1.2 implies that the bases

$$\left\{\frac{1}{R} e_{\lambda,i}\right\}_{i=1}^{q_1}, \left\{\frac{1}{R} f_{\lambda,i}\right\}_{i=1}^{q_1}, \left\{\frac{1}{R} e_{2\lambda,j}\right\}_{j=1}^{q_2}, \left\{\frac{1}{R} f_{2\lambda,j}\right\}_{j=1}^{q_2}$$

in spaces $\mathfrak{p}_\lambda, \mathfrak{k}_\lambda, \mathfrak{p}_{2\lambda}, \mathfrak{k}_{2\lambda}$ are orthonormal, so due to Lemma 2.4 one has the following generators of $\mathrm{Diff}_G(Q_{\mathbf{S}})$:

$$\begin{aligned} &D_0 = \Lambda,\, D_1 = \sum_{i=1}^{q_1} e_{\lambda,i}^2,\, D_2 = \sum_{i=1}^{q_1} f_{\lambda,i}^2,\, D_3 = \frac{1}{2}\sum_{i=1}^{q_1}\{e_{\lambda,i}, f_{\lambda,i}\}\,, \\ &D_4 = \sum_{j=1}^{q_2} e_{2\lambda,j}^2,\, D_5 = \sum_{j=1}^{q_2} f_{2\lambda,j}^2,\, D_6 = \frac{1}{2}\sum_{j=1}^{q_2}\{e_{2\lambda,j}, f_{2\lambda,j}\}\,, \end{aligned} \tag{3.2}$$

where $\{\cdot,\cdot\}$ means anticommutator. From (1.6) one easily gets:

$$\begin{aligned} &[D_0, D_1] = -D_3,\, [D_0, D_2] = D_3,\, [D_0, D_3] = \frac{1}{2}(D_1 - D_2)\,, \\ &[D_0, D_4] = -2D_6,\, [D_0, D_5] = 2D_6\,, [D_0, D_6] = D_4 - D_5\,. \end{aligned}$$

In order to find full system of generators and relations in $\mathrm{Diff}_G(Q_{\mathbf{S}})$ we need more detailed information about the Ad_{K_0}-action in $\tilde{\mathfrak{p}}$ and commutators in $\mathfrak{g}$. This information will be extracted in the following sections from the models of two-point homogeneous compact Riemannian spaces.

It is easily seen that every automorphism of Lie algebra $\mathfrak{g}$, conserving its subalgebra $\mathfrak{k}_0$, generates an automorphism of $\mathrm{Diff}_G(Q_{\mathbf{S}})$. From relations (1.3) one obtains that the map $\sigma : \mathfrak{g} \to \mathfrak{g}$, $\sigma|_{\mathfrak{k}} = \mathrm{id}$, $\sigma|_{\mathfrak{p}} = -\mathrm{id}$ is the automorphism of $\mathfrak{g}$. It generates the automorphism of $\mathrm{Diff}_G(Q_{\mathbf{S}})$: $D_0 \to -D_0$, $D_1 \to D_1$, $D_2 \to D_2$, $D_3 \to -D_3$, $D_4 \to D_4$, $D_5 \to D_5$, $D_6 \to -D_6$. We shall denote it by the same symbol σ.

Besides, the ad_Λ-action generates the one-parametric group ζ_α of automorphisms for the algebra $\mathrm{Diff}_G(Q_{\mathbf{S}})$. From (1.6) one obtains $\zeta_\alpha(\Lambda) = \Lambda$ and

$$\begin{aligned} \zeta_\alpha(e_{\lambda,i}) &= \cos(\alpha/2) e_{\lambda,i} - \sin(\alpha/2) f_{\lambda,i}\,, \\ \zeta_\alpha(f_{\lambda,i}) &= \sin(\alpha/2) e_{\lambda,i} + \cos(\alpha/2) f_{\lambda,i},\, i = 1, \dots, q_1\,, \\ \zeta_\alpha(e_{2\lambda,j}) &= \cos(\alpha) e_{2\lambda,j} - \sin(\alpha) f_{2\lambda,j}\,, \\ \zeta_\alpha(f_{2\lambda,i}) &= \sin(\alpha) e_{2\lambda,j} + \cos(\alpha) f_{2\lambda,j},\, j = 1, \dots, q_2\,. \end{aligned}$$

Therefore, it holds

$$\begin{aligned} &\zeta_\alpha(D_0) = D_0,\, \zeta_\alpha(D_1) = \cos^2(\alpha/2) D_1 + \sin^2(\alpha/2) D_2 - \sin(\alpha) D_3\,, \\ &\zeta_\alpha(D_2) = \sin^2(\alpha/2) D_1 + \cos^2(\alpha/2) D_2 + \sin\alpha D_3\,, \\ &\zeta_\alpha(D_3) = \frac{1}{2}\sin(\alpha)(D_1 - D_2) + \cos(\alpha) D_3\,, \\ &\zeta_\alpha(D_4) = \cos^2(\alpha) D_4 + \sin^2(\alpha) D_5 - \sin(2\alpha) D_6\,, \\ &\zeta_\alpha(D_5) = \sin^2(\alpha) D_4 + \cos^2(\alpha) D_5 + \sin(2\alpha) D_6\,, \\ &\zeta_\alpha(D_6) = \frac{1}{2}\sin(2\alpha)(D_4 - D_5) + \cos(2\alpha) D_6\,. \end{aligned}$$

In particular $\zeta_\pi(D_1) = D_2$, $\zeta_\pi(D_2) = D_1$, $\zeta_\pi(D_3) = -D_3$, $\zeta_\pi(D_i) = D_i$, $i = 0, 4, 5, 6$.

Due to the orthonormality of the base

$$\frac{1}{R}\Lambda,\ \frac{1}{R}e_{\lambda,i},\ \frac{1}{R}f_{\lambda,i},\ \frac{1}{R}e_{2\lambda,j},\ \frac{1}{R}f_{2\lambda,j},\ i = 1, \dots, q_1, j = 1, \dots, q_2$$

in the space $\tilde{\mathfrak{p}}$ the operator $C_1 = D_0^2 + D_1 + D_2 + D_4 + D_5$ is the Casimir one and lies in the centre of the algebra $\mathrm{Diff}_G(Q_{\mathbf{S}})$ in accordance with corollary 2.2.

For future study integrals of the classical two-body problem on two-point homogeneous spaces in Chap. 7 one needs the information on the center of the algebra $\mathrm{Diff}_G(Q_S)$. The full description of $\mathrm{ZDiff}_G(Q_S)$ seems to be a difficult problem. The general theory (see corollary 2.2) guarantees only that the set $\eta \circ \lambda^*(S(\mathfrak{g})^G)$ lies in $\mathrm{ZDiff}_G(Q_S)$. It is not known if the center of $\mathrm{Diff}_G(Q_{\mathbf{S}})$ is exhausted by $\eta \circ \lambda^*(S(\mathfrak{g})^G)$ or not. Note that in some examples below the map $\eta \circ \lambda^*|_{S(\mathfrak{g})^G}$ is not injective.

In order to find generators of $\eta \circ \lambda^*(S(\mathfrak{g})^G)$ one should firstly describe independent invariant elements of Ad_G-action in $S(\mathfrak{g})$. Their numbers and degrees for groups under consideration are given by Proposition 1.5. But their explicit form (especially for high degrees) seems to be absent in the literature. Secondly, it requires cumbersome calculations to express these invariant elements through chosen independent Ad_{K_0}-invariant elements in $S(\widetilde{\mathfrak{p}})$ modulo $(U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$. Below we shall find some elements from $\mathrm{ZDiff}_G(Q_S)$ of lower degrees by direct calculations and prove that they form a full systems of generators in algebras $\eta \circ \lambda^*(S(\mathfrak{g})^G)$ for spaces $\mathbf{P}^n(\mathbb{C})_{\mathbf{S}}, \mathbf{P}^n(\mathbb{R})_{\mathbf{S}}, \mathbf{S}^n_{\mathbf{S}}$ and their noncompact analogous.

Let $\pi_3 : Q_{\mathbf{S}} \to Q$ be the canonical projection and $\tilde{\pi}_3$ is the dual map, acting as $f \mapsto f \circ \pi_1$, where f is a function on Q. Due to the identification $\mathfrak{p} \simeq T_{x_0}Q$ it is clear that the operator $(D_0^2 + D_1 + D_2) \circ \tilde{\pi}_3$ is the Laplace-Beltrami operator on Q.

3.2 Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{H})_{\mathbf{S}})$

Here we use notations from Sect. 1.3.1.

3.2.1 Generators of Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{H})_{\mathbf{S}})$

Consider now the total space of unit spheres bundle $\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}$ over the space $\mathbf{P}^n(\mathbb{H})$. Let $(\mathbf{z}, \zeta)$ be a general point of the space $\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}$, where $\mathbf{z} \in \mathbf{P}^n(\mathbb{H})$, $\zeta \in T_{\mathbf{z}}\mathbf{P}^n(\mathbb{H})$. Due to the isomorphism $\mathbf{P}^1(\mathbb{H}) \cong \mathbf{S}^4$ we assume here $n \geqslant 2$.

Suppose that $\tilde{\mathbf{z}}_0 = (1, 0, \dots, 0) \in \mathbb{H}^{n+1}$ and an element $\xi_0 \in T_{\tilde{\mathbf{z}}_0}\mathbb{H}^{n+1} \cong \mathbb{H}^{n+1}$ has coordinates $(0, 1, 0, \dots, 0)$. Put $\mathbf{z}_0 = \pi\tilde{\mathbf{z}}_0$, $\zeta_0 = \pi_*\xi_0 \in T_{\mathbf{z}_0}\mathbf{P}^n(\mathbb{H})$.

The stationary subgroup K_0 of the group $\mathrm{U}_{\mathbb{H}}(n+1)$, corresponding to the point $(\mathbf{z}_0, \zeta_0)$, is generated by the group $K_1 = \mathrm{U}_{\mathbb{H}}(n-1)$, acting onto the last $(n-1)$ homogeneous coordinates, and by the group $K_2 = \mathrm{U}_{\mathbb{H}}(1)$, acting by the left multiplication of all homogeneous coordinates by quaternions with the unit norm. In particular the group K_0 is connected. Its $(2n^2-3n+4)$-dimensional Lie algebra $\mathfrak{k}_0$ (corresponding to Proposition 1.2) is generated by elements (1.11) with $3 \leqslant k \leqslant j \leqslant n+1$ and also by the elements

$$\sum_{k=1}^{n+1} \Upsilon_{kk}, \sum_{k=1}^{n+1} \Omega_{kk}, \sum_{k=1}^{n+1} \Theta_{kk} .$$

Suppose that the complimentary subspace $\tilde{\mathfrak{p}} \subset \mathfrak{g} = \mathrm{u}_{\mathbb{H}}(n+1)$ to the subalgebra $\mathfrak{k}_0 \subset \mathfrak{g}$ is spanned by elements

$$\begin{gathered}\Psi_{1k},\, \Upsilon_{1k},\, \Omega_{1k},\, \Theta_{1k},\, 2 \leqslant k \leqslant n+1,\, \Psi_{2k},\, \Upsilon_{2k},\, \Omega_{2k},\, \Theta_{2k},\, 3 \leqslant k \leqslant n+1\,, \\ \Upsilon_* = \frac{\mathbf{i}}{2}(E_{11}-E_{22})\,,\, \Omega_* = \frac{\mathbf{j}}{2}(E_{11}-E_{22})\,,\, \Theta_* = \frac{\mathbf{k}}{2}(E_{11}-E_{22})\ .\end{gathered} \tag{3.3}$$

Taking into account relations (1.12) it is easily obtained that the expansion $\mathfrak{u}_{\mathbb{H}}(n+1) = \tilde{\mathfrak{p}} \oplus \mathfrak{k}_0$ is reductive, i.e., $[\tilde{\mathfrak{p}}, \mathfrak{k}_0] \subset \tilde{\mathfrak{p}}$.

It is readily seen from (1.12) that setting:

$$\begin{gathered}\Lambda = -\Psi_{12},\, e_{\lambda,k-2} = \Psi_{1k},\, e_{\lambda,n-3+k} = \Upsilon_{1k}, \\ e_{\lambda,2n-4+k} = \Omega_{1k},\, e_{\lambda,3n-5+k} = \Theta_{1k}\,, \\ f_{\lambda,k-2} = -\Psi_{2k},\, f_{\lambda,n-3+k} = -\Upsilon_{2k},\, f_{\lambda,2n-4+k} = -\Omega_{2k}, \\ f_{\lambda,3n-5+k} = -\Theta_{2k},\, k = 3,\dots,n+1,\, e_{2\lambda,1} = \Upsilon_{12}\,, \\ e_{2\lambda,2} = \Omega_{12},\, e_{2\lambda,3} = \Theta_{12},\, f_{2\lambda,1} = \Upsilon_*,\, f_{2\lambda,2} = \Omega_*,\, f_{2\lambda,3} = \Theta_*\,,\end{gathered} \tag{3.4}$$

one gets the base from Proposition 1.2 for $q_1 = 4n-4, q_2 = 3$.

Now we shall find the full set of independent Ad_{K_0}-invariant elements in $S(\tilde{\mathfrak{p}})$. According to Sect. 3.1, the expansion $\tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$ is invariant w.r.t. the Ad_{K_0}-action. In the space $\mathfrak{a}$ the K_0-action is trivial that gives the invariant element $D_0 = \Lambda \in \lambda\left(S(\tilde{\mathfrak{p}})^{K_0}\right)$, already found in Sect. 3.1.

From formulas (3.4) one sees that the space $\mathfrak{p}_\lambda \cong \mathbb{H}^{n-1}$ consists of matrices of the form

$$\begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & -\bar{a}_1 & \dots & -\bar{a}_{n-1} \\ 0 & 0 & 0 & \dots & 0 \\ a_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix},\ a_1,\dots,a_{n-1} \in \mathbb{H}\ .$$

Likewise, the space $\mathfrak{k}_\lambda \cong \mathbb{H}^{n-1}$ consists of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b^* \\ 0 & b & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\bar{b}_1 & \dots & -\bar{b}_{n-1} \\ 0 & b_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n-1} & 0 & \dots & 0 \end{pmatrix},\ b_1,\dots,b_{n-1} \in \mathbb{H}\ .$$

Due to the formula

$$\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} = \begin{pmatrix} 0 & -(Ua)^* \\ Ua & 0 \end{pmatrix}, \, U \in \mathrm{U}_{\mathbb{H}}(n-1), a \in \mathbb{H}^{n-1}$$

the action of the group K_1 in the space $\mathfrak{p}_\lambda$ is equivalent to the tautological action of the group $U_{\mathbb{H}}(n-1)$ in the space $\mathbb{H}^{n-1} : a \to Ua$. In the space $\mathfrak{k}_\lambda$ the action of K_1 is similar: $b \to Ub$.

The tautological action of the group $\mathrm{U}_{\mathbb{H}}(n-1)$ in the space $\mathbb{H}^{n-1}$ has only one independent real invariant $\langle \mathbf{z}, \mathbf{z} \rangle$, $\mathbf{z} \in \mathbb{H}^{n-1}$ and the diagonal action of $\mathrm{U}_{\mathbb{H}}(n-1)$ in the space $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \cong \mathbb{H}^{n-1} \oplus \mathbb{H}^{n-1}$ has six (independent iff $n \geqslant 3$) real invariants:

$$\langle \mathbf{z}_1, \mathbf{z}_1 \rangle \in \mathbb{R}, \, \langle \mathbf{z}_2, \mathbf{z}_2 \rangle \in \mathbb{R}, \, \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \in \mathbb{H} \cong \mathbb{R}^4, \, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{H}^{n-1} \,. \tag{3.5}$$

Due to Lemma 2.4 there are corresponding elements from $\lambda\left(S(\widetilde{\mathfrak{p}})^{K_1}\right)$:

$$\begin{aligned}
D_1 &= \sum_{k=3}^{n+1} \left(\Psi_{1k}^2 + \Upsilon_{1k}^2 + \Omega_{1k}^2 + \Theta_{1k}^2\right), \, D_2 = \sum_{k=3}^{n+1} \left(\Psi_{2k}^2 + \Upsilon_{2k}^2 + \Omega_{2k}^2 + \Theta_{2k}^2\right), \\
D_3 &= -\frac{1}{2} \sum_{k=3}^{n+1} \left(\{\Psi_{1k}, \Psi_{2k}\} + \{\Upsilon_{1k}, \Upsilon_{2k}\} + \{\Omega_{1k}, \Omega_{2k}\} + \{\Theta_{1k}, \Theta_{2k}\}\right), \\
\Box_1 &= \frac{1}{2} \sum_{k=3}^{n+1} \left(-\{\Psi_{1k}, \Upsilon_{2k}\} + \{\Psi_{2k}, \Upsilon_{1k}\} + \{\Theta_{1k}, \Omega_{2k}\} - \{\Theta_{2k}, \Omega_{1k}\}\right), \quad (3.6) \\
\Box_2 &= \frac{1}{2} \sum_{k=3}^{n+1} \left(-\{\Psi_{1k}, \Omega_{2k}\} + \{\Psi_{2k}, \Omega_{1k}\} + \{\Upsilon_{1k}, \Theta_{2k}\} - \{\Upsilon_{2k}, \Theta_{1k}\}\right), \\
\Box_3 &= \frac{1}{2} \sum_{k=3}^{n+1} \left(-\{\Psi_{1k}, \Theta_{2k}\} + \{\Psi_{2k}, \Theta_{1k}\} + \{\Omega_{1k}, \Upsilon_{2k}\} - \{\Omega_{2k}, \Upsilon_{1k}\}\right) .
\end{aligned}$$

For $n = 2$ there is the unique independent relation between invariants (3.5):

$$|\langle \mathbf{z}_1, \mathbf{z}_2 \rangle|^2 = |\bar{z}_1 z_2|^2 = |z_1|^2 \, |z_2|^2 = \langle \mathbf{z}_1, \mathbf{z}_1 \rangle \langle \mathbf{z}_2, \mathbf{z}_2 \rangle, \, \mathbf{z}_1 = z_1, \mathbf{z}_2 = z_2 \in \mathbb{H} \,. \tag{3.7}$$

If we write this identity in coordinates, then we will obtain the well-known *Euler identity*, which is the key item in the proof of the *Lagrange theorem* from number theory: *every natural number can be represented as a sum of four squares.*

The elements D_1, D_2, D_3, already found in Sect. 3.1, are invariant w.r.t. the action of the whole group K_0, therefore they correspond to operators of the second order from $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_\mathbf{S})$. The elements $\Box_1, \Box_2, \Box_3$ are not invariant w.r.t. the action of the group $K_2 \cong \mathrm{U}_{\mathbb{H}}(1)$. Obviously, the K_2-action on the linear hull of elements $\Box_1, \Box_2, \Box_3$ is equivalent to the well-known action of the group $\mathrm{SO}(3) \cong \mathrm{U}_{\mathbb{H}}(1)/(1, -1)$ in the space $\mathbb{H}'$ of pure imaginary quaternions:

$$x \to qx\bar{q}, \, x \in \mathbb{H}', \, q \in \mathrm{U}_{\mathbb{H}}(1) \,, \tag{3.8}$$

after the identification $\Box_1 \leftrightarrow \mathbf{i}$, $\Box_1 \leftrightarrow \mathbf{j}$, $\Box_3 \leftrightarrow \mathbf{k}$.

The Ad_{K_2}-action on 3-dimensional spaces $\mathfrak{p}_{2\lambda}$, $\mathfrak{k}_{2\lambda}$ coincides with (3.8) after the identification $\Upsilon_{12}, \Upsilon_* \leftrightarrow \mathbf{i}$; $\Omega_{12}, \Omega_* \leftrightarrow \mathbf{j}$; $\Theta_{12}, \Theta_* \leftrightarrow \mathbf{k}$ and the Ad_{K_1}-action in these spaces is trivial. Thus, we are to find invariants of the diagonal action of the group SO(3) in the space $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. It is well known [207] that there are $6 = 9 - 3$ such independent invariants:

$$\langle \mathbf{x},\mathbf{x}\rangle,\ \langle \mathbf{y},\mathbf{y}\rangle,\ \langle \mathbf{z},\mathbf{z}\rangle,\ \langle \mathbf{x},\mathbf{y}\rangle,\ \langle \mathbf{x},\mathbf{z}\rangle,\ \langle \mathbf{z},\mathbf{y}\rangle,\ \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^3$$

and the invariant $\langle \mathbf{x},\mathbf{y},\mathbf{z}\rangle \equiv \langle \mathbf{x},\mathbf{y}\times\mathbf{z}\rangle$ algebraically connected with the first six:

$$\begin{aligned}\langle \mathbf{x},\mathbf{y},\mathbf{z}\rangle^2 = \mathbf{x}^2\mathbf{y}^2\mathbf{z}^2 + 2\langle \mathbf{x},\mathbf{y}\rangle\langle \mathbf{x},\mathbf{z}\rangle\langle \mathbf{y},\mathbf{z}\rangle - \mathbf{x}^2\langle \mathbf{y},\mathbf{z}\rangle^2 \\ - \mathbf{y}^2\langle \mathbf{x},\mathbf{z}\rangle^2 - \mathbf{z}^2\langle \mathbf{x},\mathbf{y}\rangle^2,\end{aligned} \tag{3.9}$$

where $\mathbf{y}\times\mathbf{z}$ is the standard vector product in $\mathbb{R}^3$. Relation (3.9) can be verified using the well-known formulas: $\langle \mathbf{x},\mathbf{y}\rangle^2 = \mathbf{x}^2\mathbf{y}^2 - \langle \mathbf{x}\times\mathbf{y}\rangle^2$ and $\mathbf{x}\times(\mathbf{y}\times\mathbf{x}) = \langle \mathbf{x},\mathbf{z}\rangle\mathbf{y} - \langle \mathbf{x},\mathbf{y}\rangle\mathbf{z}$.

Note that relation (3.9) can be written in the matrix form:

$$\langle \mathbf{x},\mathbf{y},\mathbf{z}\rangle^2 = \begin{vmatrix} \langle \mathbf{x},\mathbf{x}\rangle & \langle \mathbf{x},\mathbf{y}\rangle & \langle \mathbf{x},\mathbf{z}\rangle \\ \langle \mathbf{y},\mathbf{x}\rangle & \langle \mathbf{y},\mathbf{y}\rangle & \langle \mathbf{y},\mathbf{z}\rangle \\ \langle \mathbf{z},\mathbf{x}\rangle & \langle \mathbf{z},\mathbf{y}\rangle & \langle \mathbf{z},\mathbf{z}\rangle \end{vmatrix},$$

but this form is not convenient when determinant entries are noncommutative.

Lemma 2.4 gives the following invariant elements from $U(\mathfrak{g})^{K_0}$:

$$\begin{aligned} D_4 &= \Upsilon_{12}^2 + \Omega_{12}^2 + \Theta_{12}^2,\ D_5 = \Upsilon_*^2 + \Omega_*^2 + \Theta_*^2, \\ D_6 &= \frac{1}{2}\left(\{\Upsilon_{12},\Upsilon_*\} + \{\Omega_{12},\Omega_*\} + \{\Theta_{12},\Theta_*\}\right), \\ D_7 &= \frac{1}{2}\left(\{\Box_1,\Upsilon_{12}\} + \{\Box_2,\Omega_{12}\} + \{\Box_3,\Theta_{12}\}\right), \\ D_8 &= \frac{1}{2}\left(\{\Box_1,\Upsilon_*\} + \{\Box_2,\Omega_*\} + \{\Box_3,\Theta_*\}\right),\ D_9 = \Box_1^2 + \Box_2^2 + \Box_3^2, \\ D_{10} &= \Box_1\Omega_{12}\Theta_* - \Box_1\Omega_*\Theta_{12} + \Box_2\Upsilon_*\Theta_{12} - \Box_2\Upsilon_{12}\Theta_* + \Box_3\Omega_*\Upsilon_{12} \\ &\quad - \Box_3\Omega_{12}\Upsilon_* . \end{aligned} \tag{3.10}$$

Here we took into account that all three factors from every summand in the last expression pairwise commutate. Invariant elements D_4, D_5, D_6 were considered in the general situation in Sect. 3.1.

In fact, invariant elements D_7, D_8, D_9 and D_{10} do not belong to $\lambda\left(S\left(\widetilde{\mathfrak{p}}\right)^{K_0}\right)$ because they are not symmetric w.r.t. all transposition of their factors of the first degree. After complete symmetrization one can obtain invariant elements

$$\tilde{D}_k \equiv D_k + D_k^* \mod (U(\mathfrak{g})\mathfrak{k}_0)^{K_0},\ k = 7,8,9,10 .$$

from $\lambda\left(S\left(\widetilde{\mathfrak{p}}\right)^{K_0}\right)$, where D_k^* are elements from $U\left(\mathfrak{g}\right)^{K_0}$ with $\deg D_k^* < \deg D_k$. For convenience we will use elements D_k instead of $\tilde{D}_k$, $k = 7,8,9,10$.

Thus, operators $D_0, \ldots, D_{10}$ generate the algebra $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_\mathbf{S})$. The degrees of these generators are as follows:

$$\begin{aligned} &\deg(D_0) = 1,\ \deg(D_1) = \deg(D_2) = \deg(D_3) = \deg(D_4) = \deg(D_5) \\ &= \deg(D_6) = 2,\ \deg(D_7) = \deg(D_8) = 3,\ \deg(D_9) = \deg(D_{10}) = 4\ . \end{aligned} \tag{3.11}$$

In the model of the space $\mathbf{P}^n(\mathbb{H})$ we can transpose the coordinates z_1 and z_2. The operators $D_3, D_4, D_5, D_8, D_9, D_{10}$ are symmetric (invariant) w.r.t. this transposition and the operators $D_0, \Box_1, \Box_2, \Box_3, D_6, D_7$ are skew symmetric. The operators D_1 and D_2 turn into each other under this transposition.

It is easily verified that automorphisms ζ_α, σ act on $\Box_i, D_7, \ldots, D_{10}, i = 1, 2, 3$ as

$$\begin{aligned} \zeta_\alpha(\Box_i) &= \Box_i, i = 1,2,3,\ \zeta_\alpha(D_7) = \cos(\alpha)D_7 - \sin(\alpha)D_8, \\ \zeta_\alpha(D_8) &= \sin(\alpha)D_7 + \cos(\alpha)D_8\,, \zeta_\alpha(D_9) = D_9, \\ \zeta_\alpha(D_{10}) &= D_{10}, \sigma(\Box_i) = -\Box_i, i = 1,2,3, \\ \sigma(D_7) &= D_7,\ \sigma(D_8) = -D_8\,, \sigma(D_9) = D_9,\ \sigma(D_{10}) = D_{10}\ . \end{aligned}$$

Taking into account their action on other generators (see Sect. 3.1), one sees that the transposition of z_1 and z_2 is equivalent to the composition $\sigma \circ \zeta_\pi$ in the algebra $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_\mathbf{S})$.

In order to get the generators of the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{H})_\mathbf{S})$ one can use Proposition 1.5, formula (3.4) and make the formal substitution:

$$\begin{aligned} &\Lambda \to \mathbf{i}\Lambda,\ \Psi_{1k} \to \mathbf{i}\Psi_{1k},\ \Upsilon_{1k} \to \mathbf{i}\Upsilon_{1k},\ \Omega_{1k} \to \mathbf{i}\Omega_{1k},\ \Theta_{1k} \to \mathbf{i}\Theta_{1k},\ \Upsilon_{12} \to \mathbf{i}\Upsilon_{12}, \\ &\Omega_{12} \to \mathbf{i}\Omega_{12},\ \Theta_{12} \to \mathbf{i}\Theta_{12},\ \Psi_{2k} \to \Psi_{2k},\ \Upsilon_{2k} \to \Upsilon_{2k},\ \Omega_{2k} \to \Omega_{2k},\ \Theta_{2k} \to \Theta_{2k}, \\ &\Upsilon_* \to \Upsilon_*,\ \Omega_* \to \Omega_*,\ \Theta_* \to \Theta_*,\ k = 3, \ldots, n+1\ . \end{aligned}$$

This substitution produces the following substitution for the generators $D_0, \ldots, D_{10}$:

$$\begin{aligned} &D_0 \to \mathbf{i}\bar{D}_0,\ D_1 \to -\bar{D}_1,\ D_2 \to \bar{D}_2,\ D_3 \to \mathbf{i}\bar{D}_3,\ D_4 \to -\bar{D}_4,\ D_5 \to \bar{D}_5\,, \\ &D_6 \to \mathbf{i}\bar{D}_6,\ D_7 \to -\bar{D}_7,\ D_8 \to \mathbf{i}\bar{D}_8,\ D_9 \to -\bar{D}_9,\ D_{10} \to -\bar{D}_{10}\ . \end{aligned} \tag{3.12}$$

The operators $\bar{D}_0, \ldots, \bar{D}_{10}$ generate the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{H})_\mathbf{S})$.

3.2.2 Relations in Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_\mathbf{S})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{H})_\mathbf{S})$

Here we shall find independent relations in the algebra $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_\mathbf{S})$ for it's generators $D_0, \ldots, D_{10}$. In accordance with Sect. 2.1.4 they are of two types. First type consists of commutator relations, because a commutator of two differential operator of orders m_1 and m_2 is an operator of an order $m_3 \leqslant m_1 + m_2 - 1$. It gives $11(11-1)/2 = 55$ relations. Due to (3.9) for $n \geqslant 3$ the second type consists of only one independent relation of the form:

$$D_{10}^2 - D_9D_4D_5 - D_7D_6D_8 - D_8D_6D_7 + D_9D_6^2 + D_7D_5D_7 + D_8D_4D_8 = D', \tag{3.13}$$

where D' is an operator of an order $\leqslant 7$ polynomial in $D_0, \ldots, D_{10}$. The expression for it is given below. If $n = 2$ the formula (3.7) gives another independent relation of the form:

$$\frac{1}{2}\{D_1, D_2\} - D_3^2 - D_9 = D'',$$

where D'' is an operator of an order $\leqslant 3$, polynomial in $D_0, \ldots, D_8$. Direct calculations give $D'' = D_1 + D_2$, therefore in the case $n = 2$ one has the additional relation:

$$\frac{1}{2}\{D_1, D_2\} - D_3^2 - D_9 = D_1 + D_2 \ . \tag{3.14}$$

For $n = 2$ using this relation one can exclude the element D_9 from the list of generators.

In principle, all relations can be obtained by straightforward calculations in $U(\mathfrak{g})$ modulo $(U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$, but these calculations became too cumbersome to write all of them here. In appendix A there are examples of deriving some commutator relations for the algebra $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_\mathbf{S})$. After getting some commutator relations by direct calculations it is possible to find some other ones (see appendix A) using the Jacobi identity:

$$[D_i, [D_j, D_k]] + [D_k, [D_i, D_j]] + [D_j, [D_k, D_i]] = 0\,,$$

which is valid, in particular, in every associative algebra. This identity gives also a tool for checking the commutator relations already found.

The element D' can be written in the following way $D' = \sum_{i=2}^{7} \delta_i$, where

$$\begin{aligned}
\delta_7 &= D_7(D_1 - D_2)D_5 + 2D_8D_3D_4 - D_7\{D_3, D_6\} - \frac{1}{2}D_8\{D_1 - D_2, D_6\} \\
&\quad + 2D_9D_6D_0, \\
\delta_6 &= \frac{5}{4}D_0(D_1 - D_2)D_8 - \frac{5}{2}D_0D_3D_7 + D_7^2 + D_8^2 + 2D_9D_4 - \frac{9}{4}D_3^2D_4 \\
&\quad - \frac{9}{16}(D_1 - D_2)^2D_5 + \frac{9}{4}(D_1 - D_2)D_3D_6 - \frac{3}{2}(D_1 + D_2)D_5D_4 \\
&\quad + \frac{3}{2}(D_1 + D_2)D_6^2 - \frac{13}{4}D_0^2D_9 - \frac{5}{2}D_0^2D_{10} + \frac{3}{2}(D_4 + D_5)D_{10}\,, \\
\delta_5 &= \frac{33}{8}D_0(D_1 + D_2)D_6 - \frac{9}{4}\left(D_3D_8 + \frac{1}{2}(D_1 - D_2)D_7\right)\,, \\
\delta_4 &= \frac{9}{4}D_3^2 + \frac{9}{16}(D_1 - D_2)^2 + \frac{1}{8}(D_1 + D_2)(9D_5 - 15D_4) + \frac{19}{2}D_9 + \frac{3}{2}D_{10} \\
&\quad + \frac{3}{2}D_0^2(D_1 + D_2) + 3n(n-1)(D_5D_4 - D_6^2)\,, \\
\delta_3 &= -\frac{33}{4}n(n-1)D_0D_6\,, \\
\delta_2 &= \frac{1}{4}n(n-1)(15D_4 - 9D_5) - 3n(n-1)D_0^2\,.
\end{aligned}$$

Below there are all 55 commutator relations in lexicographic order.

$$
\begin{aligned}
&[D_0, D_1] = -D_3,\ [D_0, D_2] = D_3,\ [D_0, D_3] = \frac{1}{2}(D_1 - D_2),\ [D_0, D_4] = -2D_6\,,\\
&[D_0, D_5] = 2D_6,\ [D_0, D_6] = D_4 - D_5, [D_0, D_7] = -D_8,\ [D_0, D_8] = D_7,\\
&[D_0, D_9] = 0,\ [D_0, D_{10}] = 0,\ [D_1, D_2] = -\{D_0, D_3\} - 2D_7,\\
&[D_1, D_3] = -\frac{1}{2}\{D_0, D_1\} + D_8 + n(n-1)D_0\,,\\
&[D_1, D_4] = 2D_7,\ [D_1, D_5] = 0,\ [D_1, D_6] = D_8,\ [D_1, D_7] = -\frac{1}{2}\{D_3, D_6\}\\
&\qquad - \frac{1}{2}\{D_1, D_4\} + \frac{3}{8}(D_1 - D_2) + D_9 + D_{10} + n(n-1)D_4,\\
&[D_1, D_8] = -\frac{1}{2}\{D_3, D_5\} - \frac{1}{2}\{D_1, D_6\} + \frac{3}{4}D_3 + n(n-1)D_6,\\
&[D_1, D_9] = -\{D_3, D_8\} - \{D_1, D_7\} - \frac{3}{4}\{D_0, D_3\} + 2\left(n - \frac{3}{2}\right)\left(n + \frac{1}{2}\right)D_7,\\
&[D_1, D_{10}] = \frac{1}{2}\{D_6, D_8\} - \frac{1}{2}\{D_5, D_7\} + \frac{3}{8}\{D_0, D_3\} + \frac{1}{2}D_7,\\
&[D_2, D_3] = \frac{1}{2}\{D_0, D_2\} + D_8 - n(n-1)D_0, [D_2, D_4] = -2D_7,\ [D_2, D_5] = 0\,,\\
&[D_2, D_6] = -D_8,\ [D_2, D_7] = -\frac{1}{2}\{D_3, D_6\} + \frac{1}{2}\{D_2, D_4\}\\
&\qquad + \frac{3}{8}(D_1 - D_2) - D_9 - D_{10} - n(n-1)D_4,\\
&[D_2, D_8] = -\frac{1}{2}\{D_3, D_5\} + \frac{1}{2}\{D_2, D_6\} + \frac{3}{4}D_3 - n(n-1)D_6,\\
&[D_2, D_9] = -\{D_3, D_8\} + \{D_2, D_7\} + \frac{3}{4}\{D_0, D_3\}\\
&\qquad - 2\left(n - \frac{3}{2}\right)\left(n + \frac{1}{2}\right)D_7,\\
&[D_2, D_{10}] = -\frac{1}{2}\{D_6, D_8\} + \frac{1}{2}\{D_5, D_7\} - \frac{3}{8}\{D_0, D_3\}\\
&\qquad - \frac{1}{2}D_7,\ [D_3, D_4] = 0,\ [D_3, D_5] = 2D_8,\\
&[D_3, D_6] = D_7,\ [D_3, D_7] = -\frac{1}{4}\{D_1 + D_2, D_6\} + n(n-1)D_6,\\
&[D_3, D_8] = -\frac{1}{4}\{D_1 + D_2, D_5\} + n(n-1)D_5 + D_9 + D_{10},\\
&[D_3, D_9] = -\frac{1}{2}\{D_1 + D_2, D_8\} + \frac{3}{8}\{D_0, D_1 - D_2\}\\
&\qquad + 2\left(n - \frac{3}{2}\right)\left(n + \frac{1}{2}\right)D_8,\ [D_3, D_{10}] = \frac{1}{2}\{D_6, D_7\} - \frac{1}{2}\{D_4, D_8\}\\
&\qquad - \frac{3}{16}\{D_0, D_1 - D_2\} + \frac{1}{2}D_8,\ [D_4, D_5] = -2\{D_0, D_6\},\\
&[D_4, D_6] = -\{D_0, D_4\} + \frac{3}{2}D_0\,, [D_4, D_7] = \frac{1}{2}\{D_1 - D_2, D_4\} + \frac{3}{4}(D_2 - D_1),\\
&[D_4, D_8] = \frac{1}{2}\{D_1 - D_2, D_6\} - \{D_0, D_7\}\,,
\end{aligned}
\tag{3.15}
$$

$$[D_4, D_9] = \{D_1 - D_2, D_7\},\ [D_4, D_{10}] = 0,\ [D_5, D_6] = \{D_0, D_5\} - \frac{3}{2}D_0\,,$$
$$[D_5, D_7] = \{D_3, D_6\} + \{D_0, D_8\},\ [D_5, D_8] = \{D_3, D_5\} - \frac{3}{2}D_3,$$
$$[D_5, D_9] = 2\{D_3, D_8\}\,, [D_5, D_{10}] = 0,$$
$$[D_6, D_7] = \frac{1}{4}\{D_1 - D_2, D_6\} + \frac{1}{2}\{D_3, D_4\} + \frac{1}{2}\{D_0, D_7\} - \frac{3}{4}D_3\,,$$
$$[D_6, D_8] = \frac{1}{4}\{D_1 - D_2, D_5\} + \frac{1}{2}\{D_3, D_6\} - \frac{1}{2}\{D_0, D_8\} + \frac{3}{8}(D_2 - D_1)\,,$$
$$[D_6, D_9] = \frac{1}{2}\{D_1 - D_2, D_8\} + \{D_3, D_7\},\ [D_6, D_{10}] = 0,$$
$$[D_7, D_8] = \frac{1}{4}\{D_1 - D_2, D_8\} - \frac{1}{2}\{D_3, D_7\} + \frac{3}{16}\{D_0, D_1 + D_2\}$$
$$- \frac{1}{2}\{D_0, D_9 + D_{10}\} - \frac{3}{4}n(n-1)D_0,$$
$$[D_7, D_9] = \frac{1}{4}\{D_3, D_6\} + \frac{1}{8}\{D_1 - D_2, D_4\} + \frac{1}{2}\{D_1 - D_2, D_9 + D_{10}\}$$
$$- \frac{3}{8}(D_1^2 - D_2^2) + \frac{3}{4}\left(n^2 - n - \frac{1}{4}\right)(D_1 - D_2),$$
$$[D_7, D_{10}] = \frac{1}{4}\{D_2 - D_1, D_6^2\} - \frac{1}{4}\{\{D_0, D_7\}, D_6\} + \frac{1}{4}\{\{D_0, D_4\}, D_8\}$$
$$+ \frac{1}{8}\{\{D_1 - D_2, D_5\}, D_4\} - \frac{1}{4}\{D_3, D_6\}$$
$$+ \frac{1}{8}\{D_2 - D_1, 3D_4 + D_5\} - \frac{1}{2}\{D_0, D_8\} + \frac{15}{32}(D_1 - D_2)\,,$$
$$[D_8, D_9] = \frac{1}{8}\{D_1 - D_2, D_6\} + \frac{1}{4}\{D_3, D_5\} - \frac{3}{8}\{D_3, D_1 + D_2\}$$
$$+ \{D_3, D_9 + D_{10}\} + \frac{3}{2}\left(n^2 - n - \frac{1}{4}\right)D_3,$$
$$[D_8, D_{10}] = -\frac{1}{4}\{\{D_3, D_6\}, D_6\} + \frac{1}{4}\{\{D_0, D_6\}, D_8\} - \frac{1}{4}\{\{D_0, D_5\}, D_7\}$$
$$+ \frac{1}{4}\{\{D_3, D_5\}, D_4\} - \frac{1}{2}\{D_3, D_5\} - \frac{1}{4}\{D_3, D_4\} + \frac{1}{4}\{D_0, D_7\}$$
$$+ \frac{9}{16}D_3,\ [D_9, D_{10}] = \frac{1}{4}\{-\{D_6, D_8\} + \{D_5, D_7\}, D_1 - D_2\}$$
$$+ \frac{1}{2}\{\{D_3, D_8\}, D_4\} - \frac{1}{2}\{\{D_3, D_6\}, D_7\} + \frac{1}{4}\{D_2 - D_1, D_7\}$$
$$- \frac{1}{2}\{D_3, D_8\}\,.$$

It is interesting that the operators D_9 and D_{10} arise on the right hand sides of these relations only in the combination $D_9 + D_{10}$.

Using relations (3.15) it is not difficult to verify once again that the Casimir operator $C_1 = D_0^2 + D_1 + D_2 + D_4 + D_5$ lies in the centre of the algebra $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{H})_\mathbf{S})$ in accordance with Sect. 3.1. Since $\mathrm{U}_\mathbb{H}(n+1) \cong \mathrm{Sp}(2(n+1))$ is the compact real form of the group $\mathrm{Sp}(2(n+1), \mathbb{C})$, Proposition 1.5 implies that $\mathrm{Ad}_{U_{\mathbb{H}}(n+1)}$-action in $\mathfrak{u}_{\mathbb{H}}(n+1)$ has $n+1$ independent invariants of degrees $2, 4, 6, \ldots, 2n+2$.

Proposition 3.1. *The image of the map* $\eta \circ \lambda^*|_{S(\mathfrak{u}_{\mathbb{H}}(n+1))^{\mathrm{U}_{\mathbb{H}}(n+1)}}$ *is generated by some elements of degree* $\leqslant 8$.

Proof. The Lie algebra $\mathfrak{u}_{\mathbb{H}}(n')$ consists of matrices of the form $A+B\mathbf{j}$, $A, B \in \mathrm{Mat}(n', \mathbb{C})$, $\bar{A}^T = -A$, $B^T = B$. It is easy to verify that the map

$$A + B\mathbf{j} \to \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathrm{Mat}(2n', \mathbb{C}) \tag{3.16}$$

defines an isomorphism of $\mathfrak{u}_{\mathbb{H}}(n')$ with the algebra $\mathfrak{sp}(n') := \mathfrak{sp}(n', \mathbb{C}) \cap \mathfrak{u}(2n')$. Groups $U_{\mathbb{H}}(n')$ and $\mathrm{Sp}(n')$ are also isomorphic. It is known [134] that free generators of the algebra $S\left(\mathfrak{sp}(n')\right)^{\mathrm{Sp}(n')}$ correspond to coefficients p_{2k}, $k = 1, \ldots, n'$ of the polynomial:

$$p_C(\chi) = \det\left(\chi \,\mathrm{id} - \mathbf{i}C\right) = \chi^{2n'} + \sum_{k=1}^{2n'} p_k \chi^{2n'-k}, \; C \in \mathfrak{sp}(n') \, .$$

Note that coefficients p_{2k-1}, $k = 1, \ldots, n'$ vanish and p_{2k}, $k = 1, \ldots, n'$ are real. Indeed, since $C \in \mathfrak{sp}(n', \mathbb{C})$ it holds

$$C^T = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} C \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

and

$$\begin{aligned} p_C(\chi) = p_{C^T}(\chi) &= \det\left(- \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} (\chi \,\mathrm{id} + \mathbf{i}C) \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}^2 \det\left(-\chi \,\mathrm{id} - \mathbf{i}C\right) = p_C(-\chi) \end{aligned}$$

that proves the first claim. Also, it holds $\left(\overline{\mathbf{i}C}\right)^T = -\mathbf{i}\bar{C}^T = \mathbf{i}C$, since $C \in \mathfrak{u}(2n')$. Therefore, $\mathbf{i}C$ is a hermitian matrix, which proves the second claim.

Now let $n' = n+1$. Due to (3.4) coordinates in $\mathfrak{g} \cong \mathfrak{u}_{\mathbb{H}}(n+1)$, corresponding to the subspace $\widetilde{\mathfrak{p}}$, are located only in the first two rows and columns of a matrix $A + B\mathbf{j} \in \mathfrak{u}_{\mathbb{H}}(n+1)$. The isomorphism (3.16) maps these rows and columns into four rows and columns of the matrix

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} .$$

Therefore, these coordinates occur in p_{2k}, $k = 1, \ldots, n$ at most in the eight degree. The application of Lemma 2.4 completes the proof. □

Corollary 3.1. *Since there are only finite linearly independent elements in* $\mathrm{ZDiff}_I(P^n(\mathbb{H})_{\mathbf{S}})$ *of degree* $\leqslant 8$*, the map* $\eta \circ \lambda^*|_{S(\mathfrak{u}_{\mathbb{H}}(n+1))^{\mathrm{U}_{\mathbb{H}}(n+1)}}$ *is not injective for* n *large enough.*

Instead of passing from generators of $S\left(\mathfrak{u}_{\mathbb{H}}(n+1)\right)^{\mathrm{U}_{\mathbb{H}}(n+1)}$ to the corresponding element of $\mathrm{ZDiff}_I(P^n(\mathbb{H})_\mathbf{S})$ through cumbersome calculations we select elements from the algebra $\mathrm{ZDiff}_I(P^n(\mathbb{H})_\mathbf{S})$ of a degree not higher than 4. Straightforward calculations imply that these elements are linear combination of elements C_1^2,

$$
\begin{aligned}
C_2 &= \frac{1}{2}\{D_1, D_2\} - D_3^2 - D_9 - \left(n^2 - n - 1\right)(D_1 + D_2)\,, \\
C_3 &= \frac{1}{4}\{D_1 + D_2, D_4 + D_5\} + \frac{1}{4}(D_1 - D_2)^2 + D_3^2 + \frac{1}{4}(D_4 - D_5)^2 + D_6^2 + D_9 \\
&\quad - 2D_{10} + \frac{1}{4}\{D_0^2, D_1 + D_2 + D_4 + D_5\} + \frac{1}{4}D_0^4 - \left(n^2 - n - \frac{3}{2}\right)(D_4 + D_5) \\
&\quad + \left(-n^2 + n + \frac{7}{4}\right) D_0^2\,,
\end{aligned}
$$

of the forth degree and element C_1 of the second degree. Due to (3.14) it holds $C_2 = 0$ for $n = 2$.

In the case $n \geqslant 3$ the existence of two elements from $\mathrm{ZDiff}_I(P^n(\mathbb{H})_\mathbf{S})$ (independent from each other and C_1) of the forth degree could mean one of two following possibilities. The first one is the existence of an element from $\mathrm{ZDiff}_I(P^n(\mathbb{H})_\mathbf{S})$, not lying in $\eta \circ \lambda^* \left(S\left(\mathfrak{u}_{\mathbb{H}}(n+1)\right)^{\mathrm{U}_{\mathbb{H}}(n+1)}\right)$. The second possibility is that the image in $\mathrm{ZDiff}_I(P^n(\mathbb{H})_\mathbf{S})$ of some element of a degree $\geqslant 6$ from $S\left(\mathfrak{u}_{\mathbb{H}}(n+1)\right)^{\mathrm{U}_{\mathbb{H}}(n+1)}$ is of the forth degree.

Using substitution (3.12) one gets from (3.15) the commutator relations for the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{H})_\mathbf{S})$:

$$
\begin{aligned}
&[\bar D_0, \bar D_1] = \bar D_3,\ [\bar D_0, \bar D_2] = \bar D_3,\ [\bar D_0, \bar D_3] = \frac{1}{2}(\bar D_1 + \bar D_2),\ [\bar D_0, \bar D_4] = 2\bar D_6\,, \\
&[\bar D_0, \bar D_5] = 2\bar D_6,\ [\bar D_0, \bar D_6] = \bar D_4 + \bar D_5, [\bar D_0, \bar D_7] = \bar D_8, [\bar D_0, \bar D_8] = \bar D_7, \\
&[\bar D_0, \bar D_9] = 0\,, [\bar D_0, \bar D_{10}] = 0,\ [\bar D_1, \bar D_2] = -\{\bar D_0, \bar D_3\} - 2\bar D_7, \\
&[\bar D_1, \bar D_3] = -\frac{1}{2}\{\bar D_0, \bar D_1\} - \bar D_8 - n(n-1)\bar D_0\,, \\
&[\bar D_1, \bar D_4] = -2\bar D_7,\ [\bar D_1, \bar D_5] = 0,\ [\bar D_1, \bar D_6] = -\bar D_8, \\
&[\bar D_1, \bar D_7] = \frac{1}{2}\{\bar D_3, \bar D_6\} - \frac{1}{2}\{\bar D_1, \bar D_4\} - \frac{3}{8}(\bar D_1 + \bar D_2) - \bar D_9 - \bar D_{10} \\
&\qquad - n(n-1)\bar D_4,\ [\bar D_1, \bar D_8] = \frac{1}{2}\{\bar D_3, \bar D_5\} - \frac{1}{2}\{\bar D_1, \bar D_6\} - \frac{3}{4}\bar D_3 \\
&\qquad - n(n-1)\bar D_6,\ [\bar D_1, \bar D_9] = \{\bar D_3, \bar D_8\} - \{\bar D_1, \bar D_7\} + \frac{3}{4}\{\bar D_0, \bar D_3\} \\
&\qquad - 2\left(n - \frac{3}{2}\right)\left(n + \frac{1}{2}\right)\bar D_7,\ [\bar D_1, \bar D_{10}] = -\frac{1}{2}\{\bar D_6, \bar D_8\} + \frac{1}{2}\{\bar D_5, \bar D_7\} \\
&\qquad - \frac{3}{8}\{\bar D_0, \bar D_3\} - \frac{1}{2}\bar D_7\,, \\
&[\bar D_2, \bar D_3] = \frac{1}{2}\{\bar D_0, \bar D_2\} + \bar D_8 - n(n-1)\bar D_0,\ [\bar D_2, \bar D_4] = -2\bar D_7,\ [\bar D_2, \bar D_5] = 0\,,
\end{aligned}
$$

$$[\bar D_2,\bar D_6]=-\bar D_8,\ [\bar D_2,\bar D_7]=-\frac{1}{2}\{\bar D_3,\bar D_6\}+\frac{1}{2}\{\bar D_2,\bar D_4\}$$
$$+\frac{3}{8}(\bar D_1+\bar D_2)-\bar D_9-\bar D_{10}-n(n-1)\bar D_4,$$
$$[\bar D_2,\bar D_8]=-\frac{1}{2}\{\bar D_3,\bar D_5\}+\frac{1}{2}\{\bar D_2,\bar D_6\}+\frac{3}{4}\bar D_3-n(n-1)\bar D_6\,,$$
$$[\bar D_2,\bar D_9]=-\{\bar D_3,\bar D_8\}+\{\bar D_2,\bar D_7\}+\frac{3}{4}\{\bar D_0,\bar D_3\}-2\left(n-\frac{3}{2}\right)\left(n+\frac{1}{2}\right)\bar D_7\,, \tag{3.17}$$
$$[\bar D_2,\bar D_{10}]=-\frac{1}{2}\{\bar D_6,\bar D_8\}+\frac{1}{2}\{\bar D_5,\bar D_7\}-\frac{3}{8}\{\bar D_0,\bar D_3\}-\frac{1}{2}\bar D_7,$$
$$[\bar D_3,\bar D_4]=0,\ [\bar D_3,\bar D_5]=2\bar D_8\,,$$
$$[\bar D_3,\bar D_6]=\bar D_7,\ [\bar D_3,\bar D_7]=-\frac{1}{4}\{\bar D_1-\bar D_2,\bar D_6\}-n(n-1)\bar D_6,$$
$$[\bar D_3,\bar D_8]=-\frac{1}{4}\{\bar D_1-\bar D_2,\bar D_5\}-n(n-1)\bar D_5+\bar D_9+\bar D_{10},$$
$$[\bar D_3,\bar D_9]=-\frac{1}{2}\{\bar D_1-\bar D_2,\bar D_8\}+\frac{3}{8}\{\bar D_0,\bar D_1+\bar D_2\}-2\left(n-\frac{3}{2}\right)\left(n+\frac{1}{2}\right)\bar D_8,$$
$$[\bar D_3,\bar D_{10}]=\frac{1}{2}\{\bar D_6,\bar D_7\}-\frac{1}{2}\{\bar D_4,\bar D_8\}-\frac{3}{16}\{\bar D_0,\bar D_1+\bar D_2\}-\frac{1}{2}\bar D_8,$$
$$[\bar D_4,\bar D_5]=-2\{\bar D_0,\bar D_6\},\ [\bar D_4,\bar D_6]=-\{\bar D_0,\bar D_4\}-\frac{3}{2}\bar D_0\,,$$
$$[\bar D_4,\bar D_7]=\frac{1}{2}\{\bar D_1+\bar D_2,\bar D_4\}+\frac{3}{4}(\bar D_2+\bar D_1),$$
$$[\bar D_4,\bar D_8]=\frac{1}{2}\{\bar D_1+\bar D_2,\bar D_6\}-\{\bar D_0,\bar D_7\}\,,$$
$$[\bar D_4,\bar D_9]=\{\bar D_1+\bar D_2,\bar D_7\},\ [\bar D_4,\bar D_{10}]=0,\ [\bar D_5,\bar D_6]=\{\bar D_0,\bar D_5\}-\frac{3}{2}\bar D_0,$$
$$[\bar D_5,\bar D_7]=\{\bar D_3,\bar D_6\}+\{\bar D_0,\bar D_8\},\ [\bar D_5,\bar D_8]=\{\bar D_3,\bar D_5\}-\frac{3}{2}\bar D_3,$$
$$[\bar D_5,\bar D_9]=2\{\bar D_3,\bar D_8\},\ [\bar D_5,\bar D_{10}]=0,$$
$$[\bar D_6,\bar D_7]=\frac{1}{4}\{\bar D_1+\bar D_2,\bar D_6\}+\frac{1}{2}\{\bar D_3,\bar D_4\}+\frac{1}{2}\{\bar D_0,\bar D_7\}+\frac{3}{4}\bar D_3\,,$$
$$[\bar D_6,\bar D_8]=\frac{1}{4}\{\bar D_1+\bar D_2,\bar D_5\}+\frac{1}{2}\{\bar D_3,\bar D_6\}-\frac{1}{2}\{\bar D_0,\bar D_8\}-\frac{3}{8}(\bar D_1+\bar D_2)\,,$$
$$[\bar D_6,\bar D_9]=\frac{1}{2}\{\bar D_1+\bar D_2,\bar D_8\}+\{\bar D_3,\bar D_7\},\ [\bar D_6,\bar D_{10}]=0,$$
$$[\bar D_7,\bar D_8]=\frac{1}{4}\{\bar D_1+\bar D_2,\bar D_8\}-\frac{1}{2}\{\bar D_3,\bar D_7\}+\frac{3}{16}\{\bar D_0,\bar D_1-\bar D_2\}$$
$$-\frac{1}{2}\{\bar D_0,\bar D_9+\bar D_{10}\}+\frac{3}{4}n(n-1)\bar D_0\,,$$
$$[\bar D_7,\bar D_9]=-\frac{1}{4}\{\bar D_3,\bar D_6\}+\frac{1}{8}\{\bar D_1+\bar D_2,\bar D_4\}+\frac{1}{2}\{\bar D_1+\bar D_2,\bar D_9+\bar D_{10}\}$$
$$-\frac{3}{8}(\bar D_1^2-\bar D_2^2)-\frac{3}{4}\left(n^2-n-\frac{1}{4}\right)(\bar D_1+\bar D_2),$$

$$
\begin{aligned}
[\bar D_7, \bar D_{10}] &= -\frac{1}{4}\{\bar D_1 + \bar D_2, \bar D_6^2\} - \frac{1}{4}\{\{\bar D_0, \bar D_7\}, \bar D_6\} + \frac{1}{4}\{\{\bar D_0, \bar D_4\}, \bar D_8\} \\
&+ \frac{1}{8}\{\{\bar D_1 + \bar D_2, \bar D_5\}, \bar D_4\} + \frac{1}{4}\{\bar D_3, \bar D_6\} \\
&+ \frac{1}{8}\{\bar D_1 + \bar D_2, \bar D_5 - 3\bar D_4\} + \frac{1}{2}\{\bar D_0, \bar D_8\} - \frac{15}{32}(\bar D_1 + \bar D_2)\,, \\
[\bar D_8, \bar D_9] &= \frac{1}{8}\{\bar D_1 + \bar D_2, \bar D_6\} - \frac{1}{4}\{\bar D_3, \bar D_5\} - \frac{3}{8}\{\bar D_3, \bar D_1 - \bar D_2\} \\
&+ \{\bar D_3, \bar D_9 + \bar D_{10}\} - \frac{3}{2}\left(n^2 - n - \frac{1}{4}\right)\bar D_3, \\
[\bar D_8, \bar D_{10}] &= -\frac{1}{4}\{\{\bar D_3, \bar D_6\}, \bar D_6\} + \frac{1}{4}\{\{\bar D_0, \bar D_6\}, \bar D_8\} - \frac{1}{4}\{\{\bar D_0, \bar D_5\}, \bar D_7\} \\
&+ \frac{1}{4}\{\{\bar D_3, \bar D_5\}, \bar D_4\} + \frac{1}{2}\{\bar D_3, \bar D_5\} - \frac{1}{4}\{\bar D_3, \bar D_4\} + \frac{1}{4}\{\bar D_0, \bar D_7\} \\
&- \frac{9}{16}\bar D_3,\ [\bar D_9, \bar D_{10}] = \frac{1}{4}\{-\{\bar D_6, \bar D_8\} + \{\bar D_5, \bar D_7\}, \bar D_1 + \bar D_2\} \\
&+ \frac{1}{2}\{\{\bar D_3, \bar D_8\}, \bar D_4\} - \frac{1}{2}\{\{\bar D_3, \bar D_6\}, \bar D_7\} - \frac{1}{4}\{\bar D_1 + \bar D_2, \bar D_7\} \\
&+ \frac{1}{2}\{\bar D_3, \bar D_8\}\,.
\end{aligned}
$$

The analogue in $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{H})_\mathbf{S})$ for the operator C_1 is $\bar C_1 = \bar D_0^2 + \bar D_1 - \bar D_2 + \bar D_4 - \bar D_5$. For the operator C_2 and C_3 such analogs are respectively

$$
\bar C_2 = \frac{1}{2}\{\bar D_1, \bar D_2\} - \bar D_3^2 - \bar D_9 - (n^2 - n - 1)(\bar D_1 - \bar D_2)
$$

and

$$
\begin{aligned}
\bar C_3 &= \frac{1}{4}\{\bar D_1 - \bar D_2, \bar D_4 - \bar D_5\} + \frac{1}{4}(\bar D_1 + \bar D_2)^2 - \bar D_3^2 + \frac{1}{4}(\bar D_4 + \bar D_5)^2 - \bar D_6^2 \\
&- \bar D_9 + 2\bar D_{10} + \frac{1}{4}\{\bar D_0^2, \bar D_1 - \bar D_2 + \bar D_4 - \bar D_5\} + \frac{1}{4}\bar D_0^4 \\
&+ \left(n^2 - n - \frac{3}{2}\right)(\bar D_4 - \bar D_5) + \left(n^2 - n - \frac{7}{4}\right)\bar D_0^2\,.
\end{aligned}
$$

Relation (3.13) now becomes:

$$
\bar D_{10}^2 - \bar D_9 \bar D_4 \bar D_5 - \bar D_7 \bar D_6 \bar D_8 - \bar D_8 \bar D_6 \bar D_7 + \bar D_9 \bar D_6^2 + \bar D_7 \bar D_5 \bar D_7 + \bar D_8 \bar D_4 \bar D_8 = \sum_{i=2}^{7} \bar\delta_i, \tag{3.18}
$$

where

$$
\begin{aligned}
\bar\delta_7 &= \bar D_7(\bar D_1 + \bar D_2)\bar D_5 + 2\bar D_8 \bar D_3 \bar D_4 - \bar D_7\{\bar D_3, \bar D_6\} - \frac{1}{2}\bar D_8\{\bar D_1 + \bar D_2, \bar D_6\} \\
&+ 2\bar D_9 \bar D_6 \bar D_0, \\
\bar\delta_6 &= \frac{5}{4}\bar D_0(\bar D_1 + \bar D_2)\bar D_8 - \frac{5}{2}\bar D_0 \bar D_3 \bar D_7 + \bar D_7^2 - \bar D_8^2 + 2\bar D_9 \bar D_4 - \frac{9}{4}\bar D_3^2 \bar D_4 \\
&- \frac{9}{16}(\bar D_1 + \bar D_2)^2 \bar D_5 + \frac{9}{4}(\bar D_1 + D_2)\bar D_3 \bar D_6 - \frac{3}{2}(\bar D_1 - \bar D_2)\bar D_5 \bar D_4
\end{aligned}
$$

$$+\frac{3}{2}(\bar{D}_1-\bar{D}_2)\bar{D}_6^2-\frac{13}{4}\bar{D}_0^2\bar{D}_9-\frac{5}{2}\bar{D}_0^2\bar{D}_{10}+\frac{3}{2}(\bar{D}_4-\bar{D}_5)\bar{D}_{10},$$

$$\bar{\delta}_5=\frac{33}{8}\bar{D}_0(\bar{D}_1-\bar{D}_2)\bar{D}_6+\frac{9}{4}\left(\bar{D}_3\bar{D}_8-\frac{1}{2}(\bar{D}_1+\bar{D}_2)\bar{D}_7\right),$$

$$\bar{\delta}_4=-\frac{9}{4}\bar{D}_3^2+\frac{9}{16}(\bar{D}_1+\bar{D}_2)^2-\frac{1}{8}(\bar{D}_1-\bar{D}_2)(9\bar{D}_5+15\bar{D}_4)-\frac{19}{2}\bar{D}_9-\frac{3}{2}\bar{D}_{10}$$

$$+\frac{3}{2}\bar{D}_0^2(\bar{D}_1-\bar{D}_2)-3n(n-1)(\bar{D}_5\bar{D}_4-\bar{D}_6^2)\ ,$$

$$\bar{\delta}_3=\frac{33}{4}n(n-1)\bar{D}_0\bar{D}_6\ ,$$

$$\bar{\delta}_2=-\frac{1}{4}n(n-1)(15\bar{D}_4+9\bar{D}_5)+3n(n-1)\bar{D}_0^2\ .$$

In the case $n=2$ the additional relation (3.14) becomes:

$$\frac{1}{2}\{\bar{D}_1,\bar{D}_2\}-\bar{D}_3^2-\bar{D}_9=\bar{D}_1-\bar{D}_2\ . \qquad (3.19)$$

The correspondence with the compact case and Proposition 3.1 imply that for the space $\mathbf{H}^n(\mathbb{H})_{\mathbf{S}}$ the image of the map $\eta\circ\lambda^*|_{S(\mathfrak{g})^G}$ is generated by some elements of degree $\leqslant 8$.

3.3 Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}})$

Here we use notations from Sect. 1.3.2.

3.3.1 Generators of Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}})$

Consider now the space $\mathbf{P}^n(\mathbb{C})_{\mathbf{S}}$. Due to the isomorphism $\mathbf{P}^1(\mathbb{C})\cong\mathbf{S}^2$ we again assume that $n\geqslant 2$.

Suppose that $\tilde{\mathbf{z}}_0=(1,0,\ldots,0)\in\mathbb{C}^{n+1}$ and an element $\xi_0\in T_{\tilde{\mathbf{z}}_0}\mathbb{C}^{n+1}\cong\mathbb{C}^{n+1}$ has coordinates $(0,1,0,\ldots,0)$. Put $\mathbf{z}_0=\pi\tilde{\mathbf{z}}_0$, $\zeta_0=\pi_*\xi_0\in T_{\mathbf{z}_0}\mathbf{P}^n(\mathbb{C})$.

The stationary subgroup $K_0\subset G=\mathrm{SU}(n+1)$, corresponding to the point $(\mathbf{z}_0,\zeta_0)\in\mathbf{P}^n(\mathbb{C})_{\mathbf{S}}$, is generated by the group $K_1=\mathrm{SU}(n-1)$, acting onto the last $(n-1)$th coordinates and by the group $K_2=\mathrm{U}(1)$, acting onto the homogeneous coordinates in the space $\mathbf{P}^n(\mathbb{C})$ as:

$$(x_1:\ldots:x_{n+1})\to(e^{\mathbf{i}\phi}x_1:e^{\mathbf{i}\phi}x_2:e^{-2\mathbf{i}\phi}x_3:x_4:\ldots:x_{n+1})\ . \qquad (3.20)$$

This yields $\dim_{\mathbb{R}}K_0=(n-1)^2$, $K_0\cong\mathrm{U}(n-1)$. In particular, the group K_0 is connected.

The Lie algebra $\mathfrak{k}_0$ of the group K_0 is spanned by elements (1.14) as $3\leqslant k<j\leqslant n+1$ and elements:

$$\Upsilon_j-\Upsilon_3=\frac{\mathbf{i}}{2}(E_{33}-E_{jj}),\ 3<j\leqslant n+1,\ 2\Upsilon_3-\Upsilon_2=\frac{\mathbf{i}}{2}(E_{11}+E_{22}-2E_{33})\ .$$

Choose a subspace $\tilde{\mathfrak{p}} \subset \mathfrak{g} = \mathfrak{su}(n+1)$ as the linear hull of elements:

$$\Psi_{1k},\ \Upsilon_{1k},\ 2 \leqslant k \leqslant n+1,\ \Psi_{2k},\ \Upsilon_{2k},\ 3 \leqslant k \leqslant n+1,\ \Upsilon_* = \Upsilon_2\ . \tag{3.21}$$

It holds $\tilde{\mathfrak{p}} \oplus \mathfrak{k}_0 = \mathfrak{g}$. Taking into account relations (1.12) it is easily obtained that the expansion $\mathfrak{su}(n+1) = \tilde{\mathfrak{p}} \oplus \mathfrak{k}_0$ is reductive, i.e., $[\tilde{\mathfrak{p}}, \mathfrak{k}_0] \subset \tilde{\mathfrak{p}}$.

We will obtain the particular case of Proposition 1.2 for $q_1 = 2n-2, q_2 = 1$ setting:

$$\begin{aligned}&\Lambda = -\Psi_{12},\ e_{\lambda,k-2} = \Psi_{1k},\ e_{\lambda,n-3+k} = \Upsilon_{1k},\ f_{\lambda,k-2} = -\Psi_{2k},\\ &f_{\lambda,n-3+k} = -\Upsilon_{2k},\ e_{2\lambda,1} = \Upsilon_{12},\ f_{2\lambda,1} = \Upsilon_*,\ k = 3,\dots,n+1\ .\end{aligned} \tag{3.22}$$

Now we shall find the generators of the algebra $S\left(\widetilde{\mathfrak{p}}\right)^{K_0}$. The expansion $\tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda}$ is invariant w.r.t. the Ad_{K_0}-action. In the spaces $\mathfrak{a}$, $\mathfrak{p}_{2\lambda}$, $\mathfrak{k}_{2\lambda}$ the K_0-action is trivial that gives the invariants $D_0 = \Lambda$, $D_4 = \Upsilon_{12}$, $D_5 = \Upsilon_* \in \lambda\left(S\left(\widetilde{\mathfrak{p}}\right)^{K_0}\right)$. Operators D_4, D_5 are square roots of their analogs from Sect. 3.1.

From formulas (3.22) we see that the space $\mathfrak{p}_\lambda \cong \mathbb{C}^{n-1}$ consists of matrices of the form

$$\begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & -\bar{a}_1 & \dots & -\bar{a}_{n-1} \\ 0 & 0 & 0 & \dots & 0 \\ a_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix},\ a_1,\dots,a_{n-1} \in \mathbb{C}\ .$$

Similarly, the space $\mathfrak{k}_\lambda \cong \mathbb{C}^{n-1}$ consist of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b^* \\ 0 & b & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\bar{b}_1 & \dots & -\bar{b}_{n-1} \\ 0 & b_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n-1} & 0 & \dots & 0 \end{pmatrix},\ b_1,\dots,b_{n-1} \in \mathbb{C}\ .$$

The action of the group K_1 in the spaces $\mathfrak{p}_\lambda$ and $\mathfrak{k}_\lambda$ is equivalent to the tautological action of the group $\mathrm{SU}(n-1)$ in the space $\mathbb{C}^{n-1} : a \to Ua$, $U \in \mathrm{SU}(n-1)$, likewise in Sect. 3.2.1. It is easy to verify that the action (3.20) generates the K_1-actions: $a_1 \to \exp^{-3\mathbf{i}\phi} a_1$, $a_i \to \exp^{-\mathbf{i}\phi} a_i, b_1 \to \exp^{-3\mathbf{i}\phi} b_1$, $b_i \to \exp^{-\mathbf{i}\phi} b_i, i = 2,\dots,n-1$. Therefore, the K_0-action in spaces $\mathfrak{p}_\lambda$ and $\mathfrak{k}_\lambda$ is equivalent to the tautological $\mathrm{U}(n-1)$-action in $\mathbb{C}^{n-1}$.

This action has one independent real invariant $\langle \mathbf{z}, \mathbf{z} \rangle$, $\mathbf{z} \in \mathbb{C}^{n-1}$ and the diagonal action of $\mathrm{U}(n-1)$ in the space $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \cong \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$ has four (independent iff $n \geqslant 3$) real invariants:

$$\langle \mathbf{z}_1, \mathbf{z}_1 \rangle \in \mathbb{R},\ \langle \mathbf{z}_2, \mathbf{z}_2 \rangle \in \mathbb{R},\ \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \in \mathbb{C} \cong \mathbb{R}^2,\ \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^{n-1}\ . \tag{3.23}$$

Lemma 2.4 gives the corresponding elements from $\lambda\left(S\left(\widetilde{\mathfrak{p}}\right)^{K_0}\right)$ in the following form:

$$
\begin{aligned}
D_1 &= \sum_{k=3}^{n+1} \left(\Psi_{1k}^2 + \Upsilon_{1k}^2\right), \; D_2 = \sum_{k=3}^{n+1} \left(\Psi_{2k}^2 + \Upsilon_{2k}^2\right), \\
D_3 &= -\frac{1}{2} \sum_{k=3}^{n+1} \left(\{\Psi_{1k}, \Psi_{2k}\} + \{\Upsilon_{1k}, \Upsilon_{2k}\}\right), \\
\Box &= \frac{1}{2} \sum_{k=3}^{n+1} \left(-\{\Psi_{1k}, \Upsilon_{2k}\} + \{\Psi_{2k}, \Upsilon_{1k}\}\right) .
\end{aligned} \tag{3.24}
$$

In this case only operator $\Box$ is new w.r.t. Sect. 3.1.

If $n = 2$, then there is the unique independent relation between invariants (3.23):

$$
|\langle \mathbf{z}_1, \mathbf{z}_2\rangle|^2 = |\bar{z}_1 z_2|^2 = |z_1|^2\, |z_2|^2 = \langle \mathbf{z}_1, \mathbf{z}_1\rangle\langle \mathbf{z}_2, \mathbf{z}_2\rangle, \; \mathbf{z}_1 = z_1, \mathbf{z}_2 = z_2 \in \mathbb{C} . \tag{3.25}
$$

Thus, operators $D_0, \ldots, D_5, \Box$ generate the algebra $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$. The degrees of these generators are as follows:

$$
\begin{aligned}
\deg(D_0) = \deg(D_4) = \deg(D_5) = 1,\; & \deg(D_1) = \deg(D_2) \\
= \deg(D_3) = \deg(\Box) = 2 . &
\end{aligned} \tag{3.26}
$$

The operators D_3, D_4 are symmetric and the operators $D_0, \Box, D_5$ are skew symmetric w.r.t. the transposition of coordinates z_1 and z_2. The operators D_1 and D_2 turn into each other under this transposition.

In order to get the generators of the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}})$ we can use the formal substitution:

$$
\begin{aligned}
&\Lambda \to \mathbf{i}\Lambda,\; \Psi_{1k} \to \mathbf{i}\Psi_{1k},\; \Upsilon_{1k} \to \mathbf{i}\Upsilon_{1k},\; \Upsilon_{12} \to \mathbf{i}\Upsilon_{12}, \\
&\Psi_{2k} \to \Psi_{2k},\; \Upsilon_{2k} \to \Upsilon_{2k},\; \Upsilon_* \to \Upsilon_*,\; k = 3, \ldots, n+1 .
\end{aligned}
$$

This substitution produces the following substitution for the generators $D_0, \ldots,\ D_5, \Box$:

$$
\begin{aligned}
&D_0 \to \mathbf{i}\bar{D}_0,\; D_1 \to -\bar{D}_1,\; D_2 \to \bar{D}_2,\; D_3 \to \mathbf{i}\bar{D}_3, \\
&D_4 \to \mathbf{i}\bar{D}_4,\; \Box \to \mathbf{i}\bar{\Box},\; D_5 \to \bar{D}_5 .
\end{aligned} \tag{3.27}
$$

The operators $\bar{D}_0, \ldots, \bar{D}_5, \bar{\Box}$ generate the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}})$.

3.3.2 Relations in Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}})$

The commutator relations for the algebra $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ are as follows:

$$
\begin{aligned}
&[D_0, D_1] = -D_3,\ [D_0, D_2] = D_3,\ [D_0, D_3] = \frac{1}{2}(D_1 - D_2),\\
&[D_0, D_4] = -D_5,\ [D_0, D_5] = D_4\ ,[D_0, \Box] = 0,\\
&[D_1, D_2] = -\{D_0, D_3\} - \{\Box, D_4\},\ [D_1, D_3] = -\frac{1}{2}\{D_0, D_1\}\\
&\qquad + \frac{1}{2}\{\Box, D_5\} + \frac{(n-1)^2}{4}D_0,\ [D_1, D_4] = \Box,\ [D_1, D_5] = 0,\\
&[D_1, \Box] = -\frac{1}{2}\{D_1, D_4\} - \frac{1}{2}\{D_3, D_5\} + \frac{(n-1)^2}{4}D_4,\ [D_2, D_3] = \frac{1}{2}\{D_0, D_2\}\\
&\qquad + \frac{1}{2}\{\Box, D_5\} - \frac{(n-1)^2}{4}D_0,\ [D_2, D_4] = -\Box,\ [D_2, D_5] = 0,\\
&[D_2, \Box] = \frac{1}{2}\{D_2, D_4\} - \frac{1}{2}\{D_3, D_5\} - \frac{(n-1)^2}{4}D_4,\ [D_3, D_4] = 0,\\
&[D_3, D_5] = \Box,\ [D_3, \Box] = -\frac{1}{4}\{D_1 + D_2, D_5\} + \frac{(n-1)^2}{4}D_5,\ [D_4, D_5] = -D_0,\\
&\qquad [D_4, \Box] = \frac{1}{2}(D_1 - D_2),\ [D_5, \Box] = D_3\ .
\end{aligned}
$$

For $n > 2$ there are no relations of the second type. For $n = 2$ there is one relation of the second type due to (3.25):

$$
\frac{1}{2}\{D_1, D_2\} - D_3^2 - \Box^2 - \frac{1}{4}(D_0^2 + D_4^2 + D_5^2) = 0\ . \tag{3.28}
$$

Propositions 1.5 and 2.2 imply the existence of n independent generators for the algebra

$$
\mathrm{ZDiff}_I\,(\mathrm{SU}(n+1)) \cong \mathrm{Z}\,U(\mathfrak{u}(n+1))
$$

of degrees $2, 3, 4, \ldots, n+1$. The image in $\mathrm{ZDiff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ of the generator of the second degree is the Casimir operator $C_1 = D_0^2 + D_1 + D_2 + D_4^2 + D_5^2$. One can verify by direct calculations that all elements from $\mathrm{ZDiff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ of degrees $\leqslant 4$ are linear combinations of elements C_1, C_1^2, C_2 and C_3, where

$$
\begin{aligned}
C_2 &= (D_1 - D_2)\,D_5 - 2D_3D_4 + 2D_0\Box,\\
C_3 &= \frac{1}{2}\{D_1, D_2\} - D_3^2 - \Box^2 - \frac{n^2 - 2n - 1}{4}(D_1 + D_2)\ .
\end{aligned}
$$

Due to (3.28) in the case $n = 2$ one has $C_1 = 4C_3$. The degree of C_2 is 3 and the degree of C_3 for $n \geqslant 3$ is 4.

Proposition 3.2. *The image of the map* $\eta \circ \lambda^*|_{S(\mathfrak{u}(n+1))^{\mathrm{U}(n+1)}}$ *is generated by elements of degree* $\leqslant 4$.

Proof. It is known [134] that free generators of the algebra $S\,(\mathfrak{u}(n+1))^{\mathrm{U}(n+1)}$ correspond to coefficients p_k, $k = 1, \ldots, n+1$ of the polynomial:

$$
p_A(\chi) = \det\,(\chi\,\mathrm{id} - \mathbf{i}A) = \chi^{n+1} + \sum_{k=1}^{n+1} p_k \chi^{n+1-k},\ A \in \mathfrak{u}(n+1)\ .
$$

By the same arguments as in the proof of Proposition 3.1 coefficients p_k, $k = 1, \ldots, n{+}1$ are real. Due to (3.22) coordinates in $\mathfrak{g} \cong \mathfrak{u}(n{+}1)$, corresponding to the subspace $\widetilde{\mathfrak{p}}$, are located only in the first two rows and columns of a matrix $A \in \mathfrak{u}(n{+}1)$. Therefore, these coordinates occur in p_k, $k = 1, \ldots, n{+}1$ at most in the fourth degree. The application of Lemma 2.4 completes the proof. □

Corollary 3.2. *Elements* C_1, C_2 *and* C_3 *generate a subalgebra in* $\mathrm{ZDiff}_I(\mathbf{P}^n(\mathbb{C})_\mathbf{S})$ *containing the subalgebra* $\eta \circ \lambda^* \left(S\left(\mathfrak{u}(n+1)\right)^{\mathrm{U}(n+1)} \right)$. *Since there are only finite linearly independent elements in* $\mathrm{ZDiff}_I(\mathbf{P}^n(\mathbb{C})_\mathbf{S})$ *of degree* $\leqslant 4$, *the map* $\eta \circ \lambda^*|_{S(\mathfrak{u}(n+1))^{\mathrm{U}(n+1)}}$ *is not injective for* n *large enough.*

Using substitution (3.27) one gets analogous relations for the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{C})_\mathbf{S})$.

The commutator relations are now as follows:

$$
\begin{aligned}
&[\bar D_0, \bar D_1] = \bar D_3, [\bar D_0, \bar D_2] = \bar D_3, [\bar D_0, \bar D_3] = \frac{1}{2}(\bar D_2 + \bar D_1), [\bar D_0, \bar D_4] = \bar D_5, [\bar D_0, \bar D_5] = \bar D_4,\\
&[\bar D_0, \bar\Box] = 0,\ [\bar D_1, \bar D_2] = -\{\bar D_0, \bar D_3\} - \{\bar\Box, \bar D_4\},\ [\bar D_1, \bar D_3] = -\frac{1}{2}\{\bar D_0, \bar D_1\} - \frac{1}{2}\{\bar\Box, \bar D_5\}\\
&-\frac{(n-1)^2}{4}\bar D_0,\ [\bar D_1, \bar D_4] = -\bar\Box,\ [\bar D_1, \bar D_5] = 0,\ [\bar D_1, \bar\Box] = -\frac{1}{2}\{\bar D_1, \bar D_4\} + \frac{1}{2}\{\bar D_3, \bar D_5\}\\
&-\frac{(n-1)^2}{4}\bar D_4,\ [\bar D_2, \bar D_3] = \frac{1}{2}\{\bar D_0, \bar D_2\} + \frac{1}{2}\{\bar\Box, \bar D_5\} - \frac{(n-1)^2}{4}\bar D_0,\ [\bar D_2, \bar D_4] = -\bar\Box,\\
&[\bar D_2, \bar D_5] = 0,\ [\bar D_2, \bar\Box] = \frac{1}{2}\{\bar D_2, \bar D_4\} - \frac{1}{2}\{\bar D_3, \bar D_5\} - \frac{(n-1)^2}{4}\bar D_4,\ [\bar D_3, \bar D_4] = 0,\\
&[\bar D_3, \bar D_5] = \bar\Box,\ [\bar D_3, \bar\Box] = -\frac{1}{4}\{\bar D_1 - \bar D_2, \bar D_5\} - \frac{(n-1)^2}{4}\bar D_5,\ [\bar D_4, \bar D_5] = -\bar D_0,\\
&[\bar D_4, \bar\Box] = \frac{1}{2}(\bar D_1 + \bar D_2),\ [\bar D_5, \bar\Box] = \bar D_3\ .
\end{aligned}
$$

For $n > 2$ there are no relations of the second type. On the other hand for $n = 2$ there is one relation of the second type analogous to (3.28):

$$
\frac{1}{2}\{\bar D_1, \bar D_2\} - \bar D_3^2 - \bar\Box^2 - \frac{1}{4}(\bar D_0^2 + \bar D_4^2 - \bar D_5^2) = 0\ . \tag{3.29}
$$

The analogs in $\mathrm{ZDiff}_I(\mathbf{H}^n(\mathbb{C})_\mathbf{S})$ of operators C_1, C_2 and C_3 are

$$
\begin{aligned}
\bar C_1 &= \bar D_0^2 + \bar D_1 - \bar D_2 + \bar D_4^2 - \bar D_5^2,\\
\bar C_2 &= \left(\bar D_1 + \bar D_2\right)\bar D_5 - 2\bar D_3 \bar D_4 + 2\bar D_0 \bar\Box,\\
\bar C_3 &= \frac{1}{2}\{\bar D_1, \bar D_2\} - \bar D_3^2 - \bar\Box^2 - \frac{n^2 - 2n - 1}{4}(\bar D_1 - \bar D_2)\ .
\end{aligned}
$$

The correspondence with the compact case and corollary 3.2 imply

Corollary 3.3. *Let* G *be the identity component of the isometry group for the space* $\mathbf{H}^n(\mathbb{C})$. *Elements* $\bar C_1, \bar C_2$ *and* $\bar C_3$ *generate a subalgebra in* $\mathrm{ZDiff}_I(\mathbf{H}^n(\mathbb{C})_\mathbf{S})$ *containing the subalgebra* $\eta \circ \lambda^* \left(S\left(\mathfrak{g}\right)^G \right)$. *Since there are only finite linearly independent elements in* $\mathrm{ZDiff}_I(\mathbf{H}^n(\mathbb{C})_\mathbf{S})$ *of degree* $\leqslant 4$, *the map* $\eta \circ \lambda^*|_{S(\mathfrak{g})^G}$ *is not injective for* n *large enough.*

3.4 Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{R})_\mathbf{S})$, $\mathrm{Diff}_I(\mathbf{S}^n_\mathbf{S})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_\mathbf{S})$

Here we use notations from Sect. 1.3.3.

3.4.1 Generators of Algebras $\mathrm{Diff}_I(\mathbf{S}^n_\mathbf{S})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_\mathbf{S})$

Consider the space $\mathbf{S}^n_\mathbf{S}$. Suppose that $\tilde{\mathbf{z}}_0 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ and an element $\xi_0 \in T_{\tilde{\mathbf{z}}_0}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ has coordinates $(0, 1, 0, \dots, 0)$. Put $\mathbf{z}_0 = \pi\tilde{\mathbf{z}}_0$, $\zeta_0 = \pi_*\xi_0 \in T_{\mathbf{z}_0}\mathbf{S}^n_\mathbf{S}$.

The stationary subgroup K_0 of the group $\mathrm{SO}(n+1)$, corresponding to the point $(\mathbf{z}_0, \zeta_0) \in \mathbf{S}^n_\mathbf{S}$, is the group $\mathrm{SO}(n-1)$, acting onto the last $(n-1)$th coordinates.

The group $\mathrm{SO}(n+1)$ is a group covering of the identity component G of the isometry group for $\mathbf{P}^n(\mathbb{R})$. The group $K_0 = \mathrm{SO}(n-1) \subset \mathrm{SO}(n+1)$ is a group covering of the corresponding subgroup $K_0' \subset G$. Due to Proposition 1.6 orbits of Ad_{K_0} and $\mathrm{Ad}_{K_0'}$ actions on $\mathfrak{p} \subset \mathfrak{g}$ coincide with each other and the construction from Sect. 2.1 implies the isomorphism $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{R})_\mathbf{S}) \cong \mathrm{Diff}_I(\mathbf{S}^n_\mathbf{S})$.

The Lie algebra $\mathfrak{k}_0$ of the group K_0 is spanned by the elements Ψ_{kj} as $3 \leqslant k < j \leqslant n+1$. Choose the complimentary subspace $\tilde{\mathfrak{p}}$ to the subalgebra $\mathfrak{k}_0$ in the algebra $\mathfrak{g} = \mathfrak{so}(n+1)$ as the linear hull of elements:

$$\Psi_{1k},\ 2 \leqslant k \leqslant n+1,\ \Psi_{2k},\ 3 \leqslant k \leqslant n+1\ . \tag{3.30}$$

Then the expansion $\mathfrak{so}(n+1) = \tilde{\mathfrak{p}} \oplus \mathfrak{so}(n-1)$ is reductive.

We will obtain the particular case of Proposition 1.2 for $q_1 = 0, q_2 = n-1$ setting

$$\Lambda = -2\Psi_{12},\ e_{2\lambda,k-2} = 2\Psi_{1k},\ f_{2\lambda,k-2} = -2\Psi_{2k},\ k = 3, \dots, n+1\ . \tag{3.31}$$

Consider the expansion $\tilde{\mathfrak{p}} = \mathfrak{a} \oplus \mathfrak{k}_{2\lambda} \oplus \mathfrak{p}_{2\lambda}$, which is invariant w.r.t. the Ad_{K_0}-action. It is easy to see that the K_0-action is trivial in the space $\mathfrak{a}$ and in the spaces $\mathfrak{k}_{2\lambda}$ and $\mathfrak{p}_{2\lambda}$ it is equivalent to the tautological action of the group $\mathrm{SO}(n-1)$ in the space $\mathbb{R}^{n-1}$. The trivial K_0-action in the space $\mathfrak{a}$ has the invariant element $D_0 = \Lambda$. The description of base K_0-invariants in the space $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ is different in cases $n = 2$, $n = 3$ and $n \geqslant 4$.

The Case $n \geqslant 4$

The $\mathrm{SO}(n-1)$-action in $\mathbb{R}^{n-1}$ has one independent real invariant: $\langle \mathbf{z}, \mathbf{z} \rangle$, $\mathbf{z} \in \mathbb{R}^{n-1}$, and the diagonal action of $\mathrm{SO}(n-1)$ in the space $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda} \cong \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ has three independent real invariants:

$$\langle \mathbf{z}_1, \mathbf{z}_1 \rangle,\ \langle \mathbf{z}_2, \mathbf{z}_2 \rangle,\ \langle \mathbf{z}_1, \mathbf{z}_2 \rangle,\ \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{n-1}\ . \tag{3.32}$$

Lemma 2.4 gives the corresponding elements from $\lambda\left(S\left(\widetilde{\mathfrak{p}}\right)^{K_0}\right)$ in the following form:

$$D_1 = 4\sum_{k=3}^{n+1} \Psi_{1k}^2,\ D_2 = 4\sum_{k=3}^{n+1} \Psi_{2k}^2,\ D_3 = -2\sum_{k=3}^{n+1} \{\Psi_{1k}, \Psi_{2k}\}\ .$$

All these invariants were found in Sect. 3.1 for the general situation.

Thus, operators D_0, D_1, D_2, D_3 generate the algebra $\mathrm{Diff}_I(\mathbf{S}^n_\mathbf{S})$. The degrees of these generators are as follows:

$$\deg(D_0) = 1,\ \deg(D_1) = \deg(D_2) = \deg(D_3) = 2\ . \tag{3.33}$$

The operator D_3 is symmetric and the operators D_0 is skew symmetric w.r.t. the transposition of coordinates z_1 and z_2. The operators D_1 and D_2 turn into each other under this transposition.

The Case $n = 2$

In this case K_0 is the trivial group and the independent invariants are D_0, $D_1 = e_{2\lambda,1}$, $D_2 = f_{2\lambda,1}$. Thus, the algebra $\mathrm{Diff}_I(\mathbf{S}^2_\mathbf{S})$ is isomorphic to $U(\mathfrak{so}(3))$. The centre of this algebra is generated by the operator $D_0^2+D_1^2+D_2^2$.

The Case $n = 3$

In this case $K_0 = \mathfrak{so}(2)$ and one has the additional (with respect to the case $n \geqslant 4$) invariant of the second order

$$\Box = 2(\{\Psi_{13}, \Psi_{24}\} - \{\Psi_{14}, \Psi_{23}\})\ .$$

It is algebraically connected with operators D_0, D_1, D_2, D_3 that are defined as in the case $n \geqslant 4$.

Generators of the Algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_\mathbf{S})$

First, let $n \geqslant 4$. In order to get the generators of the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_\mathbf{S})$ one can use the formal substitution:

$$\Lambda \to \mathbf{i}\Lambda,\ \Psi_{1k} \to \mathbf{i}\Psi_{1k},\ \Psi_{2k} \to \Psi_{2k},\ k = 3, \ldots, n+1\ .$$

This substitution produces the following substitution for the generators $D_0, \ldots, D_3$:

$$D_0 \to \mathbf{i}\bar{D}_0,\ D_1 \to -\bar{D}_1,\ D_2 \to \bar{D}_2,\ D_3 \to \mathbf{i}\bar{D}_3\ . \tag{3.34}$$

The operators $\bar{D}_0, \ldots, \bar{D}_3$ generate the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_\mathbf{S})$.

In the case $n = 3$ one has the additional substitution $\Box \to \mathbf{i}\bar{\Box}$ and the operators $\bar{D}_0, \ldots, \bar{D}_3, \bar{\Box}$ generate the algebra $\mathrm{Diff}_I(\mathbf{H}^3(\mathbb{R})_\mathbf{S})$.

In the case $n = 2$ one gets the substitutions

$$D_0 \to \mathbf{i}\bar{D}_0,\ D_1 \to \mathbf{i}\bar{D}_1,\ D_2 \to \bar{D}_2\ .$$

The algebra $\mathrm{Diff}_I(\mathbf{H}^2(\mathbb{R})_\mathbf{S})$ is isomorphic to $U(\mathfrak{so}(1,2))$ and its centre is generated by the operator $D_0^2 + D_1^2 - D_2^2$.

3.4.2 Relations in Algebras $\mathrm{Diff}_I(\mathbf{S}^n_\mathbf{S})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_\mathbf{S})$

Here we shall consider only the case $n \geqslant 3$, since $\mathrm{Diff}_I(\mathbf{S}^2_\mathbf{S}) \cong U(\mathfrak{so}(3))$ and $\mathrm{Diff}_I(\mathbf{H}^2(\mathbb{R})_\mathbf{S}) \cong U(\mathfrak{so}(1,2))$.

The commutator relations for the algebra $\mathrm{Diff}_I(\mathbf{S}^n_\mathbf{S})$ are as follows:

$$\begin{aligned}
&[D_0, D_1] = -2D_3,\ [D_0, D_2] = 2D_3,\ [D_0, D_3] = D_1 - D_2,\\
&[D_1, D_2] = -2\{D_0, D_3\}\ ,\\
&[D_1, D_3] = -\{D_0, D_1\} + \frac{(n-1)(n-3)}{2} D_0,\\
&[D_2, D_3] = \{D_0, D_2\} - \frac{(n-1)(n-3)}{2} D_0\ .
\end{aligned} \tag{3.35}$$

For $n = 3$ the additional operator $\Box$ lies in the centre of the algebra $\mathrm{Diff}_I(\mathbf{S}^3_\mathbf{S})$.

For $n > 3$ there are no relations of the second type. For $n = 3$ there is one independent relation of the second type:

$$\frac{1}{2}\{D_1, D_2\} - D_0^2 = D_3^2 + \Box^2\ . \tag{3.36}$$

It is easy to verify by direct calculation that the operators $C_1 = D_0^2 + D_1 + D_2$ and

$$C_2 = \frac{1}{2}\{D_1, D_2\} - D_3^2 + \left(1 - \frac{(n-3)(n-1)}{4}\right)(D_1 + D_2)$$

lie in the centre of the algebra $\mathrm{Diff}_I(\mathbf{S}^n_\mathbf{S})$ and for $n \geqslant 4$ any operator from $\mathrm{ZDiff}_I(\mathbf{S}^n_\mathbf{S})$ of a degree $\leqslant 6$ is a polynomial in operators C_1 and C_2. If $n = 3$ it holds $C_2 = \Box^2 + C_1$ due to (3.36). Every operator from $\mathrm{ZDiff}_I(\mathbf{S}^3_\mathbf{S})$ of a degree $\leqslant 6$ is a polynomial in operators C_1 and $\Box$.

Proposition 3.3. *The image of the map* $\eta \circ \chi^*|_{S(\mathfrak{so}(n+1))^{\mathrm{SO}(n+1)}}$ *is generated by elements of degree* $\leqslant 4$.

Proof. For a matrix $A \in \mathfrak{so}(n+1)$ consider the polynomial

$$p_A(\chi) = \det(\chi\,\mathrm{id} - \mathbf{i}A) = \chi^{n+1} + \sum_{k=1}^{n+1} p_k \chi^{n+1-k}.$$

By the same arguments as in the proof of Proposition 3.1 coefficients p_k, $k = 1, \ldots, n+1$ are real. Moreover

$$p_A(\chi) = p_{A^T}(\chi) = \det(\chi\,\mathrm{id} + \mathbf{i}A) = (-1)^{n+1}\det(-\chi\,\mathrm{id} - \mathbf{i}A) = (-1)^{n+1} p_A(-\chi)\ .$$

This gives

$$\sum_{k=1}^{n+1} p_k \chi^{n+1-k} \equiv \sum_{k=1}^{n+1} (-1)^k p_k \chi^{n+1-k}$$

and coefficients p_k with odd indices vanish .

It is known [134] that free generators of $S\left(\mathfrak{so}(n+1)\right)^{\mathrm{SO}(n+1)}$ in the case $n+1 = 2\ell + 1$, $\ell \geqslant 1$ correspond to coefficients p_k, $k = 2, 4, \ldots, 2\ell$ and in the case $n+1=2\ell$, $\ell \geqslant 2$ they correspond to coefficients p_k, $k=2,4,\dots,2\ell-2$ and also to the *Pfaffian* $\operatorname{Pf} A$ of the matrix A .

The Pfaffian is given by the formula

$$\operatorname{Pf} A = \sum_{(i_1 i_2|\dots|i_{2\ell-1} i_{2\ell})} \operatorname{sgn}(i_1,\dots,i_{2\ell}) a_{i_1 i_2} \dots a_{i_{2\ell-1} i_{2\ell}},\ A \in \mathfrak{so}(2\ell),$$

where $a_{i_1 i_2}$ are elements of the matrix A and the summation is taken over all nonequivalent partitions of the set $1,2,\dots,2\ell$ into pairs (two partitions different only in ordering of pairs and in ordering inside pairs are considered to be equivalent).

Due to (3.31) coordinates in $\mathfrak{g} \cong \mathfrak{so}(n+1)$, corresponding to the subspace $\widetilde{\mathfrak{p}}$, are located only in the first two rows and columns of a matrix $A \in \mathfrak{so}(n+1)$. Therefore, these coordinates occur in p_k, $k=2,4,\dots$ and $\operatorname{Pf} A$ at most in the fourth degree.

The application of Lemma 2.4 completes the proof. □

Corollary 3.4. *Elements C_1, C_2 in the case $n \geqslant 4$ (or elements $C_1, \square$ in the case $n=3$) generate a subalgebra in $\mathrm{ZDiff}_I(\mathbf{S}^n_{\mathbf{S}})$, containing the subalgebra $\eta \circ \lambda^*(S(\mathfrak{so}(n+1))^{\mathrm{SO}(n+1)})$. Since there are only finite linearly independent elements in $\mathrm{ZDiff}_I(\mathbf{S}^n_{\mathbf{S}})$ of degree $\leqslant 4$, the map $\eta \circ \lambda^*|_{S(\mathfrak{so}(n+1))^{\mathrm{SO}(n+1)}}$ is not injective for n large enough.*

Using substitution (3.34) one gets analogous relations for the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_{\mathbf{S}})$.

The commutator relation are now as follows:

$$[\bar{D}_0,\bar{D}_1] = 2\bar{D}_3,\ [\bar{D}_0,\bar{D}_2] = 2\bar{D}_3,\ [\bar{D}_0,\bar{D}_3] = \bar{D}_1 + \bar{D}_2,\ [\bar{D}_1,\bar{D}_2] = -2\{\bar{D}_0,\bar{D}_3\},$$
$$[\bar{D}_1,\bar{D}_3] = -\{\bar{D}_0,\bar{D}_1\} - \frac{(n-1)(n-3)}{2}\bar{D}_0,$$
$$[\bar{D}_2,\bar{D}_3] = \{\bar{D}_0,\bar{D}_2\} - \frac{(n-1)(n-3)}{2}\bar{D}_0,$$

and for $n=3$ also

$$[\bar{D}_0,\bar{\square}] = [\bar{D}_1,\bar{\square}] = [\bar{D}_2,\bar{\square}] = [\bar{D}_3,\bar{\square}] = 0\ .$$

The first three relations were found in [146], but the other relations were not calculated there.

For $n>3$ there are no relations of the second type. Contrary, for $n=3$ there is one independent relation of the second type analogous to (3.36):

$$\frac{1}{2}\{\bar{D}_1,\bar{D}_2\} - \bar{D}_0^2 = \bar{D}_3^2 + \bar{\square}^2. \tag{3.37}$$

The operators $\bar{C}_1 = \bar{D}_0^2 + \bar{D}_1 - \bar{D}_2$ and

$$\bar{C}_2 = \frac{1}{2}\{\bar{D}_1,\bar{D}_2\} - \bar{D}_3^2 + \left(1 - \frac{(n-3)(n-1)}{4}\right)(\bar{D}_1 - \bar{D}_2)$$

lie in the centre of the algebra $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{R})_{\mathbf{S}})$ and for $n = 3$ it holds $\bar{C}_2 = \bar{\Box}^2 + \bar{C}_1$ due to (3.37).

For $n \geqslant 4$ every operator from $\mathrm{ZDiff}_I(\mathbf{H}^n_{\mathbf{S}})$ of a degree $\leqslant 6$ is a polynomial in operators $\bar{C}_1$ and $\bar{C}_2$. Every operator from $\mathrm{ZDiff}_I(\mathbf{H}^3_{\mathbf{S}})$ of a degree $\leqslant 6$ is a polynomial in operators $\bar{C}_1$ and $\bar{\Box}$.

The correspondence with the compact case and corollary 3.4 imply

Corollary 3.5. *Let G be the identity component of the isometry group for the space $\mathbf{H}^n(\mathbb{R})$. Elements $\bar{C}_1, \bar{C}_2$ in the case $n \geqslant 4$ (or elements $\bar{C}_1, \bar{\Box}$ in the case $n = 3$) generate a subalgebra in $\mathrm{ZDiff}_I(\mathbf{H}^n(\mathbb{R})_{\mathbf{S}})$, containing the subalgebra $\eta \circ \lambda^* \left(S(\mathfrak{g})^G\right)$. Since there are only finite linearly independent elements in $\mathrm{ZDiff}_I(\mathbf{H}^n(\mathbb{R})_{\mathbf{S}})$ of degree $\leqslant 4$, the map $\eta \circ \lambda^*|_{S(\mathfrak{g})^G}$ is not injective for n large enough.*

3.5 Algebras $\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{C}a)_{\mathbf{S}})$ and $\mathrm{Diff}_I(\mathbf{H}^n(\mathbb{C}a)_{\mathbf{S}})$

Here we use notations from Sect. 1.4.

3.5.1 Generators of Algebras $\mathrm{Diff}_I\left(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}\right)$ and $\mathrm{Diff}_I\left(\mathbf{H}^2(\mathbb{C}a)_{\mathbf{S}}\right)$

Now we are to specify the construction from Sect. 3.1 for the space $M = \mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}$.

The special base in $\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$

It is easily seen that

$$[Y_1(\xi), E_1] = 0,\ [Y_2(\xi), E_1] = X_2(\xi),\ [Y_3(\xi), E_1] = -X_3(\xi),\ \xi \in \mathbb{C}a,$$

so one can identify the space $T_{E_1}\mathbf{P}^2(\mathbb{C}a)$ with the space $(Y_2(\xi) + Y_3(\eta)|\, \xi, \eta \in \mathbb{C}a) \subset \mathfrak{m}_0$. From (1.41) one gets that the expansion

$$(Y_2(\xi) + Y_3(\eta)|\, \xi, \eta \in \mathbb{C}a) = (Y_3(\xi)|\, \xi \in \mathbb{R}) \oplus (Y_2(\xi)|\, \xi \in \mathbb{C}a) \oplus (Y_3(\xi)|\, \xi \in \mathbb{C}a')$$

is Ad_{K_0}-invariant. Therefore, in accordance with Sects. 1.2 and 3.1, denote:

$$\mathfrak{a} := (Y_3(\xi)|\, \xi \in \mathbb{R})\,,\ \mathfrak{p}_\lambda := (Y_2(\xi)|\, \xi \in \mathbb{C}a)\,,\ \mathfrak{p}_{2\lambda} := (Y_3(\xi)|\, \xi \in \mathbb{C}a')\ .$$

Choose the point y from Sect. 3.1 as $y = \left(E_1, \frac{1}{2}X_3(1)\right) \in \mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}$, where $\frac{1}{2}X_3(1) \in \mathbf{S}_{E_1}$. One has the following expansions $T_y\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}} = T_{E_1}\mathbf{P}^2(\mathbb{C}a) \oplus T_{\frac{1}{2}X_3(1)}\mathbf{S}_{E_1}$ and

$$T_{\frac{1}{2}X_3(1)}\mathbf{S}_{E_1} \simeq \{X_2(\xi)|\, \xi \in \mathbb{C}a\} \oplus \{X_3(\xi)|\, \xi \in \mathbb{C}a'\}\ .$$

Due to $\mathrm{ad}\, Y_1(\xi)\,(X_3(1)) = -X_2(\bar{\xi}),\ \xi \in \mathbb{C}a$ the space $(X_2(\xi)|\, \xi \in \mathbb{C}a) \subset T_{\frac{1}{2}X_3(1)}\mathbf{S}_{E_1}$ is identified with the space $(Y_1(\xi)|\, \xi \in \mathbb{C}a) \subset \mathfrak{m}_0$. According to the equality

$$\dim_{\mathbb{R}}(Y_1(\xi)|\, \xi \in \mathbb{C}a) = 8 = \dim_{\mathbb{R}} \mathfrak{p}_\lambda$$

denote $\mathfrak{k}_\lambda := (Y_1(\xi)|\, \xi \in \mathbb{C}a)$. Thus, one gets $\mathfrak{m}_0 = \mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_\lambda$.

Denote by $A_{ij} \in \mathfrak{k}'$, $i \neq j$ the generators of the rotation in the 2-dimensional plane, containing elements $e_i, e_j \in \mathbb{C}a_3$ such that $A_{ij}e_j = e_i$, $A_{ij}e_i = -e_j$. The operators A_{ij}, $1 \leqslant i < j \leqslant 7$ compose a base of the algebra $\mathfrak{k}_0$. Similar to the quaternion case the linear hull $\mathfrak{q} \subset \mathfrak{k}'$ of elements $A_{0\alpha} =: A_\alpha$, $\alpha = 1, \ldots, 7$ is Ad_{K_0}-invariant and is identified through the K_0-action on $T_y\mathbf{P}^2(\mathbb{C}a)_\mathbf{S}$ with the space $(X_3(\xi)|\, \xi \in \mathbb{C}a') \subset T_{\frac{1}{2}X_3(1)}\mathbf{S}_{E_1}$. Therefore, we denote $\mathfrak{k}_{2\lambda} := \mathfrak{q}$.

Lemma 3.1. *It holds*

$$A^{(1)}_\alpha = L_\alpha,\ A^{(2)}_\alpha = R_\alpha,\ A^{(1)}_{\alpha\beta} = L_{\beta,\alpha},\ A^{(2)}_{\alpha\beta} = R_{\beta,\alpha},$$
$$C_{3,e_\alpha,e_\beta} = 4A_{\beta,\alpha},\ C_{3,e_0,e_\alpha} = -4A_\alpha,\ \alpha, \beta = 1, \ldots, 7, \alpha \neq \beta\ .$$

Proof. From (1.40) one has

$$A^{(3)}(\xi) = \overline{A^{(1)}(\bar\xi) + \bar\xi A^{(2)}(1)}\ .$$

Let $A^{(1)} = L_\alpha$, then $A^{(2)} = R_\alpha$ and $A^{(3)}(e_k) = \frac{1}{2}\overline{(e_\alpha\bar e_k + \bar e_k e_\alpha)} = -\frac{1}{2}(e_k e_\alpha + e_\alpha e_k)$. If $1 \leqslant k \neq \alpha$, then $e_k e_\alpha = -e_\alpha e_k$ and $A^{(3)}(e_k) = 0$. Therefore, $A^{(3)} = A_\alpha$ due to $A^{(3)}(1) = -e_\alpha$, $A^{(3)}(e_\alpha) = 1$. This proves $A^{(1)}_\alpha = L_\alpha$, $A^{(2)}_\alpha = R_\alpha$.

Let now $A^{(1)} = L_{\beta,\alpha}$, then $A^{(2)} = R_{\beta,\alpha}$ and $A^{(3)}(e_k) = \frac{1}{2}\overline{(e_\beta \cdot e_\alpha\bar e_k + \bar e_k \cdot e_\alpha e_\beta)} = \frac{1}{2}\,(e_k e_\alpha \cdot e_\beta + e_\beta e_\alpha \cdot e_k)$. It is easy to verify by direct computation that if $\alpha = 1, \beta = 2$, then $A^{(3)}(e_k) = 0$, for $k \neq 1, 2$ and $A^{(3)}(e_1) = -e_2$, $A^{(3)}(e_2) = e_1$. Thus $L^{(3)}_{\beta,\alpha} = A_{12}$. Therefore $L^{(3)}_{\beta,\alpha} = A_{\alpha\beta}$ for any other pair of e_α, e_β, since the group G_2 of automorphisms of $\mathbb{C}a$ acts transitively on all pairs of imaginary units ([142], lecture 15). This proves $A^{(1)}_{\alpha\beta} = L_{\beta,\alpha}$, $A^{(2)}_{\alpha\beta} = R_{\beta,\alpha}$.

The last two equalities of this lemma are obvious. □

Let summarize these reasoning in the following proposition:

Proposition 3.4. *Choose the following base*

$$\Lambda := \frac{1}{2}\,\mathrm{ad}\,Y_3(e_1),\ e_{2\lambda,\alpha} := \frac{1}{2}\,\mathrm{ad}\,Y_3(e_\alpha),\ f_{2\lambda,\alpha} := \varkappa A_\alpha,$$
$$e_{\lambda,i} := -\frac{1}{2}\,\mathrm{ad}\,Y_2(\bar e_i),\ f_{\lambda,i} := \frac{1}{2}\,\mathrm{ad}\,Y_1(e_i),\ \tilde A_{\alpha\beta} := \varkappa A_{\alpha\beta}$$

in the space $\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$, where latin indices vary from 0 to 7 and greek ones (except λ) vary from 1 to 7. Then one gets the following commutator relations:

$$[\Lambda, e_{2\lambda,\alpha}] = -f_{2\lambda,\alpha},\ [\Lambda, f_{2\lambda,\alpha}] = e_{2\lambda,\alpha},\ [\Lambda, e_{\lambda,i}] = -\frac{1}{2}f_{\lambda,i},$$
$$[\Lambda, f_{\lambda,i}] = \frac{1}{2}e_{\lambda,i},\ [\Lambda, \tilde A_{\alpha\beta}] = 0, [e_{2\lambda,\alpha}, e_{2\lambda,\beta}] = \tilde A_{\beta\alpha},$$
$$[e_{2\lambda,\alpha}, f_{2\lambda,\beta}] = -\delta_{\alpha\beta}\Lambda,\ [f_{2\lambda,\alpha}, f_{2\lambda,\beta}] = \tilde A_{\beta\alpha},\ [e_{2\lambda,\alpha}, e_{\lambda,j}] = \frac{1}{2}f_{\lambda,e_\alpha e_j},$$

$$[e_{2\lambda,\alpha}, f_{\lambda,j}] = \frac{1}{2}e_{\lambda,e_\alpha e_j}\,,\ [f_{2\lambda,\alpha}, e_{\lambda,j}] = -\frac{1}{2}e_{\lambda,e_\alpha e_j},\ [f_{2\lambda,\alpha}, f_{\lambda,j}] = \frac{1}{2}f_{\lambda,e_\alpha e_j}\,,$$
$$[e_{\lambda,i}, e_{\lambda,j}] = \frac{1}{4}\varkappa C_{2,\bar{e}_i,\bar{e}_j} = \frac{1}{2}f_{2\lambda,e_i\bar{e}_j} + \frac{1}{2}\varkappa\tilde{C}_{2,i,j},\ i \neq j\,,$$
$$[f_{\lambda,i}, f_{\lambda,j}] = \frac{1}{4}\varkappa C_{1,e_i,e_j} = -\frac{1}{2}f_{2\lambda,e_i\bar{e}_j} + \frac{1}{2}\varkappa\tilde{C}_{1,i,j},\ i \neq j\,,$$
$$[e_{\lambda,i}, f_{\lambda,j}] = \begin{cases} -\frac{1}{2}\Lambda,\ i = j \\ -\frac{1}{2}e_{2\lambda,e_i\bar{e}_j},\ i \neq j \end{cases},$$

where $f_{\lambda,e_\alpha e_j} := f_{\lambda,i}$ *if* $e_\alpha e_j = e_i$ *and* $f_{\lambda,e_\alpha e_j} := -f_{\lambda,i}$ *if* $e_\alpha e_j = -e_i$. *Analogous notation we use for* $e_{\lambda,i}, e_{2\lambda,\gamma}, f_{2\lambda,\gamma}$. *Here operators* $\widetilde{C}_{l,i,j}$, $l = 1, 2, i \neq j$ *belongs to* $\mathfrak{k}_0$ *and act as:*

$$\widetilde{C}_{1,i,j}(e_k) = e_k e_i \cdot \bar{e}_j,\ e_k \neq 1, \pm e_i\bar{e}_j,\ \widetilde{C}_{1,i,j}(e_k) = 0,\ e_k = 1, \pm e_i\bar{e}_j,$$
$$\widetilde{C}_{2,i,j}(e_k) = e_j \cdot \bar{e}_i e_k,\ e_k \neq 1, \pm e_i\bar{e}_j,\ \widetilde{C}_{2,i,j}(e_k) = 0,\ e_k = 1, \pm e_i\bar{e}_j\ .$$

The chosen bases $\Lambda, e_{\lambda,i}, e_{2\lambda,\alpha}, f_{\lambda,i}, f_{2\lambda,\alpha}$ *in spaces* $\mathfrak{a}, \mathfrak{p}_\lambda, \mathfrak{p}_{2\lambda}, \mathfrak{k}_\lambda, \mathfrak{k}_{2\lambda}$ *correspond to notations of proposition* 1.2.

Proof. The commutator relations are consequences of (1.41), (1.42), Lemma 3.1 and relations in the algebra $\mathfrak{k}' \simeq \mathfrak{so}(8)$. For example, let us calculate the commutator $[f_{2\lambda,\alpha}, e_{\lambda,j}]$. Actually, from (1.41) and Lemma 3.1 one gets:

$$[f_{2\lambda,\alpha}, e_{\lambda,j}] = -\left[\varkappa A_\alpha, \frac{1}{2}\,\mathrm{ad}\,Y_2(\bar{e}_j)\right] = -\frac{1}{2}\,\mathrm{ad}\,Y_2\left(A^{(2)}_\alpha\bar{e}_j\right) = -\frac{1}{2}\,\mathrm{ad}\,Y_2\left(R_\alpha\bar{e}_j\right)$$
$$= -\frac{1}{4}\,\mathrm{ad}\,Y_2\left(\bar{e}_j e_\alpha\right) = \frac{1}{4}\,\mathrm{ad}\,Y_2\left(\bar{e}_j\bar{e}_\alpha\right) = -\frac{1}{2}e_{\lambda,e_\alpha e_j}\ .$$

Similar calculations are also valid for $[f_{2\lambda,\alpha}, f_{\lambda,j}]$.

Now, let us calculate $[e_{\lambda,i}, e_{\lambda,j}]$, $i \neq j$. From (1.42) we obtain:

$$[e_{\lambda,i}, e_{\lambda,j}] = \frac{1}{4}\left[\mathrm{ad}\,Y_2(\bar{e}_i), \mathrm{ad}\,Y_2(\bar{e}_j)\right] = \frac{1}{4}\varkappa C_{2,\bar{e}_i,\bar{e}_j},\ i \neq j\ .$$

From (1.32) and (1.43) one has

$$\frac{1}{2}C_{2,\bar{e}_i,\bar{e}_j}(e_k) = \frac{1}{2}\left(e_j \cdot \bar{e}_i e_k - e_i \cdot \bar{e}_j e_k\right) = -e_i \cdot \bar{e}_j e_k\ .$$

In particular,

$$\frac{1}{2}C_{2,\bar{e}_i,\bar{e}_j}(1) = -e_i\bar{e}_j,\ \frac{1}{2}C_{2,\bar{e}_i,\bar{e}_j}(e_i\bar{e}_j) = -\left(e_i \cdot \bar{e}_j\right)^2 = 1,$$

so

$$\frac{1}{2}\varkappa C_{2,\bar{e}_i,\bar{e}_j} = \varkappa A_{e_i\bar{e}_j} + \varkappa\tilde{C}_{2,i,j} = f_{2\lambda,e_i\bar{e}_j} + \varkappa\tilde{C}_{2,i,j},$$

where $\tilde{C}_{2,i,j} \in \mathfrak{k}_0$ and

$$\tilde{C}_{2,i,j}(e_k) = e_j \cdot \bar{e}_i e_k,\ e_k \neq 1, \pm e_i\bar{e}_j,\ \tilde{C}_{2,i,j}(e_i\bar{e}_j) = \tilde{C}_{2,i,j}(1) = 0\ .$$

The similar calculations are also valid for $[f_{\lambda,i}, f_{\lambda,j}]$. □

Invariant elements in $S(\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$

Invariant operators $D_0, \dots, D_6$, corresponding to some Ad_{K_0}-invariant elements in $S(\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$, are already constructed in Sect. 3.1. Here we shall construct other independent Ad_{K_0}-invariant elements in $S(\mathfrak{a} \oplus \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$ (or equivalently in $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$, since $\mathfrak{a}$ is an invariant one-dimensional space) and also corresponding invariant differential operators.

An element $\Phi \in K'$ is from $K_0 \subset K'$ iff $\Phi_3(1) = 1$ and then $\Phi_3(\xi) = \xi$ for any $\xi \in \mathbb{R} \subset \mathbb{C}a_3$. Below in this sections $\Phi \in K_0$. The orthogonality of Φ_i means that

$$\mathrm{Re}\left(\Phi_i(\xi)\overline{\Phi_i(\eta)}\right) = \mathrm{Re}(\xi\bar{\eta}),\ \xi, \eta \in \mathbb{C}a_i\ . \tag{3.38}$$

In particular, $\Phi_i(\xi)\overline{\Phi_i(\xi)} = |\xi|^2$ and

$$\overline{\Phi_i(\xi)^{-1}} = \Phi_i(\xi)/|\xi|^2\ . \tag{3.39}$$

For $\eta = \bar{\xi}$ from (1.39) one gets $\Phi_1(\xi)\Phi_2(\bar{\xi}) = \overline{\Phi_3(|\xi|^2)} = |\xi|^2$, so (3.39) implies $\Phi_1(\xi) = |\xi|^2\Phi_2(\bar{\xi})^{-1} = \overline{\Phi_2(\bar{\xi})}$ and

$$\Phi_1 = \imath \circ \Phi_2 \circ \imath\ . \tag{3.40}$$

Let $Q_1(\xi, \eta) := \mathrm{Re}(\xi\eta)$, $\xi \in \mathbb{C}a_1, \eta \in \mathbb{C}a_2$. From (3.38) and (3.40) we get:

$$Q_1(\Phi_1(\xi), \Phi_2(\eta)) = \mathrm{Re}(\Phi_1(\xi)\Phi_2(\eta)) = \mathrm{Re}(\Phi_1(\xi)\overline{\Phi_1(\bar{\eta})}) = \mathrm{Re}(\xi\bar{\bar{\eta}}) = Q_1(\xi, \eta)\ .$$

Thus, the form $Q_1(\xi, \eta)$ is invariant w.r.t. the K_0-action.

From Propositions 1.8 and 1.9 it follows that $\Phi_1 = g^L, \Phi_2 = g^R, \Phi_3 = \imath \circ g^V \circ \imath = g^V$, where g^L, g^R, g^V are respectively left spinor, right spinor and vector representation of the group $K_0 \simeq \mathrm{Spin}(7)$, since $\imath|_{\mathbb{C}a_3'} = -\mathrm{id}$. Besides, the K_0-action on $\mathrm{Im}(\xi\eta)$, $\xi \in \mathbb{C}a_1, \eta \in \mathbb{C}a_2$ equals g^V, so the form $Q_2(\xi, \eta, \zeta) := \mathrm{Re}\left(\mathrm{Im}(\xi\eta)\zeta\right)$ is invariant under K_0-action for $\zeta \in \mathbb{C}a_3'$.

Due to the identification $X_2(\xi)$ with $Y_2(\xi)$ in accordance with the formula $[Y_2(\xi), E_1] = X_2(\xi)$ one can identify the K_0-action on $\mathfrak{p}_\lambda^* \cong \mathbb{C}a_2$ with operators Φ_2.[1] Similarly, due to formulas $\mathrm{ad}\, Y_1(\xi)\, (X_3(1)) = -X_2(\bar{\xi})$, $[Y_3(\xi), E_1] = -X_3(\xi)$ one can identify the K_0-action on $\mathfrak{k}_\lambda^* \cong \mathbb{C}a_1$ with operators $\imath \circ \Phi_2 \circ \imath = \Phi_1$ and K_0-action on $\mathfrak{p}_{2\lambda}^* \cong \mathbb{C}a_3'$ with operators Φ_3. The K_0-action on $\mathfrak{k}_{2\lambda}^*$ is equivalent to the Φ_3-action on another examples of $\mathbb{C}a_3'$. Any invariant of K_0-action on $\mathbb{C}a_1 \oplus \mathbb{C}a_2 \oplus \mathbb{C}a_3' \oplus \mathbb{C}a_3'$ is naturally identified with K_0-invariant elements from

$$S\left((\mathbb{C}a_1 \oplus \mathbb{C}a_2 \oplus \mathbb{C}a_3' \oplus \mathbb{C}a_3')^*\right) \cong S\left(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}\right)\ .$$

Therefore, the analogue of the invariant $\mathrm{Im}(\xi\eta)$, $\xi \in \mathbb{C}a_1, \eta \in \mathbb{C}a_2$ is

$$\sum_{i \neq j} f_{\lambda,i}^* e_{\lambda,\bar{e}_j}^* \otimes e_i e_j \in S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}) \otimes \mathbb{C}a,$$

[1] See the footnote on page 20.

where $e^*_{\lambda,i}, f^*_{\lambda,i}$ are elements from $S\,(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$, corresponding to elements $e_{\lambda,i}, f_{\lambda,i} \in \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$, like in Sect. 2.1.2.

Thus, the invariant Q_2 corresponds to the following invariant element from the algebra $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$

$$\sum_{i\neq j} f^*_{\lambda,i} e^*_{\lambda,\bar{e}_j} e^*_{2\lambda,e_i e_j} = \sum_{i\neq j} f^*_{\lambda,i} e^*_{\lambda,j} e^*_{2\lambda,e_i \bar{e}_j} \,.$$

Therefore, one can define the invariant differential operator:

$$D_7 = -\frac{1}{4}\sum_{i\neq j} \left\{ \{ f_{\lambda,i}, e_{\lambda,j}\}, e_{2\lambda,e_i\bar{e}_j} \right\} = \frac{1}{4}\sum_{i\neq j} \left\{ \{ f_{\lambda,j}, e_{\lambda,i}\}, e_{2\lambda,e_i\bar{e}_j} \right\},$$

since for $i \neq j$, it holds $e_j\bar{e}_i = -\overline{e_j\bar{e}_i} = -e_i\bar{e}_j$.

Substitution $e_{2\lambda,e_i\bar{e}_j} \to f_{2\lambda,e_i\bar{e}_j}$ gives the invariant differential operator:

$$D_8 = -\frac{1}{4}\sum_{i\neq j} \left\{ \{ f_{\lambda,i}, e_{\lambda,j}\}, f_{2\lambda,e_i\bar{e}_j} \right\} = \frac{1}{4}\sum_{i\neq j} \left\{ \{ f_{\lambda,j}, e_{\lambda,i}\}, f_{2\lambda,e_i\bar{e}_j} \right\} \,.$$

It is clear that (1.39) remains valid after the cyclic permutation of indices $1, 2, 3$ since the definition of the group K' is symmetric w.r.t. this permutation. Therefore, one gets

$$\Phi_3(\zeta)\Phi_1(\xi) = \overline{\Phi_2(\overline{\zeta\xi})},\ \Phi_2(\eta)\Phi_3(\zeta) = \overline{\Phi_1(\overline{\eta\zeta})},\ \xi \in \mathbb{C}a_1, \eta \in \mathbb{C}a_2, \zeta \in \mathbb{C}a'_3 \,. \tag{3.41}$$

Define the function

$$P(\xi, \eta, \zeta_1, \zeta_2) := \mathrm{Re}(\zeta_1\xi \cdot \eta\zeta_2),\ \xi \in \mathbb{C}a_1, \eta \in \mathbb{C}a_2, \zeta_1, \zeta_2 \in \mathbb{C}a'_3 \,.$$

It is invariant w.r.t. the K_0-shifts, since due to (3.41), (3.40) and (3.38) it holds:

$$\begin{aligned} P\left(\Phi_1(\xi), \Phi_2(\eta), \Phi_3(\zeta_1), \Phi_3(\zeta_2)\right) &= \mathrm{Re}\left(\Phi_3(\zeta_1)\Phi_1(\xi) \cdot \Phi_2(\eta)\Phi_3(\zeta_2)\right) \\ &= \mathrm{Re}\left(\overline{\Phi_2\left(\overline{\zeta_1\xi}\right)}\ \overline{\Phi_1\left(\overline{\eta\zeta_2}\right)}\right) \\ &= \mathrm{Re}\left(\Phi_1\left(\zeta_1\xi\right) \overline{\Phi_1\left(\overline{\eta\zeta_2}\right)}\right) \\ &= \mathrm{Re}\left(\zeta_1\xi \cdot \overline{\overline{\eta\zeta_2}}\right) = P(\xi, \eta, \zeta_1, \zeta_2) \,. \end{aligned}$$

Functions $P(\xi, \eta, \zeta_1, \zeta_2)$ and $P(\xi, \eta, \zeta_2, \zeta_1)$ depend from each other and invariants of the second order. Indeed, the corollary 15.12 in [2] gives:

$$\mathrm{Re}(ab{\cdot}c) = \mathrm{Re}(bc{\cdot}a) = \mathrm{Re}(ca{\cdot}b) = \mathrm{Re}(a{\cdot}bc) = \mathrm{Re}(b{\cdot}ca) = \mathrm{Re}(c{\cdot}ab),\ a, b, c \in \mathbb{C}a.$$

Therefore, using the Moufang identity (1.31), one gets

$$\begin{aligned} P(\xi, \eta, \zeta, \zeta) &= \mathrm{Re}(\zeta \cdot \xi\eta \cdot \zeta) = \mathrm{Re}(\zeta^2 \cdot \xi\eta) \\ &= -\,\mathrm{Re}(|\zeta|^2\xi\eta) = -|\zeta|^2\,\mathrm{Re}(\xi\eta) = -|\zeta|^2 Q_1(\xi, \eta) \,. \end{aligned} \tag{3.42}$$

This means that for $\zeta_1 = \zeta_2 = \zeta$ the invariant $P(\xi, \eta, \zeta, \zeta)$ is expressed through invariants of the second order. Using the polarization of (3.42) w.r.t. ζ, i.e., the substitution $\zeta = \zeta_1 + \zeta_2$, we get:

$$P(\xi, \eta, \zeta_1, \zeta_2) + P(\xi, \eta, \zeta_2, \zeta_1) = -2\langle \zeta_1, \zeta_2 \rangle Q_1(\xi, \eta).$$

This implies the dependence of two invariants $P(\xi, \eta, \zeta_1, \zeta_2)$, $P(\xi, \eta, \zeta_2, \zeta_1)$ and the invariants $Q_1(\xi, \eta)$, $\langle \zeta_1, \zeta_2 \rangle$ of the second order. The last two invariants correspond to operators D_3 and D_6.

For constructing the invariant differential operator D_9 we shall use the invariant function

$$P(\xi, \eta, \zeta_1, \zeta_2) - P(\xi, \eta, \zeta_2, \zeta_1) .$$

Using $\sum_k e_{\lambda, \bar{e}_k} \otimes e_k$ as the analog of η one gets the corresponding expression from $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$:

$$\sum_{\substack{i \neq j \\ j \neq k}} \left(f^*_{2\lambda, e_j \bar{e}_i} f^*_{\lambda, i} e^*_{\lambda, \bar{e}_k} e^*_{2\lambda, \bar{e}_k \bar{e}_j} - e^*_{2\lambda, e_j \bar{e}_i} f^*_{\lambda, i} e^*_{\lambda, \bar{e}_k} f^*_{2\lambda, \bar{e}_k \bar{e}_j} \right)$$
$$= \sum_{\substack{i \neq j \\ j \neq k}} \left(e^*_{2\lambda, e_i \bar{e}_j} f^*_{\lambda, i} e^*_{\lambda, k} f^*_{2\lambda, e_k \bar{e}_j} - f^*_{2\lambda, e_i \bar{e}_j} f^*_{\lambda, i} e^*_{\lambda, k} e^*_{2\lambda, e_k \bar{e}_j} \right)$$

again due to $e_j \bar{e}_i = -e_i \bar{e}_j$ for $i \neq j$.

Define the corresponding invariant differential operator as

$$D_9 = \frac{1}{8} \sum_{\substack{i \neq j \\ j \neq k}} \left(\left\{ \left\{ e_{2\lambda, e_i \bar{e}_j}, f_{\lambda, i} \right\}, \left\{ f_{2\lambda, e_k \bar{e}_j}, e_{\lambda, k} \right\} \right\} \right.$$
$$\left. - \left\{ \left\{ e_{2\lambda, e_i \bar{e}_j}, e_{\lambda, i} \right\}, \left\{ f_{2\lambda, e_k \bar{e}_j}, f_{\lambda, k} \right\} \right\} \right) .$$

Let us show that there are exactly 9 independent K_0-invariants in $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$.

Indeed, it holds $\dim(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}) = 8 + 8 + 7 + 7 = 30$ and $\dim K_0 = \dim \mathrm{Spin}(7) = 21$. Therefore, the codimension of K_0-orbits in the space $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ is at least $30 - 21 = 9$ and there should be at least 9 independent K_0-invariants in the algebra $S(\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda})$.

From another hand, it is obvious that the stationary subgroup of the group Spin(7), acting in the space $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ by the representation $g^V \oplus g^V$, that corresponds to a point in general position is Spin(5). Therefore, the dimension of general Spin(7)-orbits in $\mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ is $\dim \mathrm{Spin}(7) - \dim \mathrm{Spin}(5) = 11$. The group Spin(5) is isomorphic to $U_{\mathbb{H}}(2)$, see [2], proposition 5.1. In Sect. 3.2.1 the six independent invariants of the diagonal $U_{\mathbb{H}}(2)$-action in $\mathbb{H}^2 \oplus \mathbb{H}^2 \simeq \mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda$ were found, so general orbits of the last action are 10-dimensional, since $\dim_{\mathbb{R}}(\mathbb{H}^2 \oplus \mathbb{H}^2) - 6 = 10$. Thus, general Spin(7)-orbits in $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$ are $11 + 10 = 21$-dimensional, their codimension is 9 and there are exactly 9 functionally independent invariants of Spin(7)-action in $\mathfrak{p}_\lambda \oplus \mathfrak{k}_\lambda \oplus \mathfrak{p}_{2\lambda} \oplus \mathfrak{k}_{2\lambda}$.

It is not known if there are any other invariants of this action, which are polynomial in $e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,\alpha}, f_{2\lambda,\alpha}$ and are not polynomial in $D_0, \dots, D_9$. Such invariants if exist should be connected with $D_0, \dots, D_9$ by an algebraic nonlinear equation. In the case of $\mathbf{P}^n(\mathbb{H})_\mathbf{S}$, $n \geqslant 3$ there is such invariant D_{10} and its square D_{10}^2 is polynomial in $D_0, \dots, D_9$. The operator D_{10} arises in commutator relations of $D_1, \dots, D_9$.

In the next section it will be found that all commutators of operators $D_0, \dots, D_9$ in the octonionic case are polynomial in $D_0, \dots, D_9$. Therefore, it seems to be probable in the octonionic case that there is no an analogue of the operator D_{10}, independent on $D_0, \dots, D_9$.

The rigorous proof of the completeness of the system of invariants $D_0, \cdots, D_9$ seems to be a difficult problem as many problems in invariant theory [202]. In any case operators $D_0, \cdots, D_9$ generate the subalgebra of $\mathrm{Diff}_I\left(\mathbf{P}^2(\mathbb{C}a)_\mathbf{S}\right)$.

It is easily verified that automorphisms ζ_α, σ act on D_7, D_8, D_9 as

$$\begin{aligned}
&\zeta_\alpha(D_7) = \cos(\alpha)D_7 - \sin(\alpha)D_8,\ \zeta_\alpha(D_8) = \sin(\alpha)D_7 + \cos(\alpha)D_8,\\
&\zeta_\alpha(D_9) = D_9,\ \sigma(D_7) = D_7,\ \sigma(D_8) = -D_8, \sigma(D_9) = D_9.
\end{aligned}$$

Similarly to the previous sections, in order to get the generators of the algebra $\mathrm{Diff}_I(\mathbf{H}^2(\mathbb{C}a)_\mathbf{S})$ one can use Proposition 1.5 and make the formal substitution:

$$\Lambda \to \mathbf{i}\Lambda, e_{\lambda,i} \to \mathbf{i}e_{\lambda,i}, f_{\lambda,i} \to f_{\lambda,i}, e_{2\lambda,\alpha} \to \mathbf{i}e_{2\lambda,\alpha}, f_{2\lambda,\alpha} \to f_{2\lambda,\alpha}.$$

This substitution produces the corresponding substitution for the generators $D_0, \dots, D_9$:

$$\begin{aligned}
&D_0 \to \mathbf{i}\bar{D}_0,\ D_1 \to -\bar{D}_1,\ D_2 \to \bar{D}_2,\ D_3 \to \mathbf{i}\bar{D}_3,\ D_4 \to -\bar{D}_4,\\
&D_5 \to \bar{D}_5,\ D_6 \to \mathbf{i}\bar{D}_6,\ D_7 \to -\bar{D}_7,\ D_8 \to \mathbf{i}\bar{D}_8,\ D_9 \to -\bar{D}_9\ .
\end{aligned} \tag{3.43}$$

The operators $\bar{D}_0, \dots, \bar{D}_9$ generate the algebra $\mathrm{Diff}_I(\mathbf{H}^2(\mathbb{C}a)_\mathbf{S})$.

3.5.2 Relations in Algebras $\mathrm{Diff}_I\left(\mathbf{P}^2(\mathbb{C}a)_\mathbf{S}\right)$ and $\mathrm{Diff}_I\left(\mathbf{H}^2(\mathbb{C}a)_\mathbf{S}\right)$

Below there are all 45 commutator relations for operators $D_0, \dots, D_9$. An example of calculation of such relation is in appendix A. All methods described in Sect. 3.2.2 for calculating commutator relations were used in this case. Besides, the numeration of the base elements $e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,\alpha}, f_{2\lambda,\alpha}$ by octonionic units e_i, $i = 0, \dots, 7$ is very convenient.

$$\begin{aligned}
&[D_0, D_1] = -D_3,\ [D_0, D_2] = D_3,\ [D_0, D_3] = \frac{1}{2}(D_1 - D_2),\ [D_0, D_4] = -2D_6,\\
&[D_0, D_5] = 2D_6,\ [D_0, D_6] = D_4 - D_5, [D_0, D_7] = -D_8, [D_0, D_8] = D_7,\\
&[D_0, D_9] = 0,\ [D_1, D_2] = -\{D_0, D_3\} - 2D_7,\ [D_1, D_3] = -\frac{1}{2}\{D_0, D_1\} + D_8\\
&\qquad + 10D_0,\ [D_1, D_4] = 2D_7,\ [D_1, D_5] = 0,\ [D_1, D_6] = D_8,\\
&[D_1, D_7] = \frac{1}{2}\{D_1, D_2 - D_4\} - D_9 - \frac{1}{2}\{D_3, D_6\}\\
&\qquad - D_3^2 - 5D_0^2 - \frac{3}{32}D_1 - \frac{283}{32}D_2 + \frac{19}{2}D_4 - \frac{1}{2}D_5,
\end{aligned}$$

$$
\begin{aligned}
&[D_1, D_8] = -\frac{1}{2}\{D_3, D_5\} - \frac{1}{2}\{D_1, D_6\} + 10D_6 + \frac{35}{4}D_3,\\
&[D_1, D_9] = \frac{1}{2}\{D_5, D_7\} - \frac{1}{2}\{D_6, D_8\} - \frac{189}{32}\{D_0, D_3\} - \frac{169}{16}D_7,\\
&[D_2, D_3] = \frac{1}{2}\{D_0, D_2\} + D_8 - 10D_0,\ [D_2, D_4] = -2D_7,\ [D_2, D_5] = 0,\\
&[D_2, D_6] = -D_8,\ [D_2, D_7] = -\frac{1}{2}\{D_2, D_1 - D_4\} + D_9 - \frac{1}{2}\{D_3, D_6\}\\
&\qquad + D_3^2 + 5D_0^2 + \frac{3}{32}D_2 + \frac{283}{32}D_1 - \frac{19}{2}D_4 + \frac{1}{2}D_5,\\
&[D_2, D_8] = \frac{1}{2}\{D_2, D_6\} - \frac{1}{2}\{D_3, D_5\} + \frac{35}{4}D_3 - 10D_6,\\
&[D_2, D_9] = -\frac{1}{2}\{D_5, D_7\} + \frac{1}{2}\{D_6, D_8\} + \frac{189}{32}\{D_0, D_3\} + \frac{169}{16}D_7, [D_3, D_4] = 0,\\
&[D_3, D_5] = 2D_8,\ [D_3, D_6] = D_7,\ [D_3, D_7] = -\frac{1}{4}\{D_1 + D_2, D_6\} + 10D_6,\\
&[D_3, D_8] = \frac{1}{2}\{D_1, D_2\} - \frac{1}{4}\{D_1 + D_2, D_5\} - D_9 - D_3^2 - 5D_0^2 - \frac{143}{32}(D_1 + D_2)\\
&\qquad - \frac{1}{2}D_4 + \frac{19}{2}D_5,\ [D_3, D_9] = \frac{1}{2}\{D_4, D_8\} - \frac{1}{2}\{D_6, D_7\}\\
&\qquad + \frac{189}{64}\{D_0, D_1 - D_2\} - \frac{169}{16}D_8,\ [D_4, D_5] = -2\{D_0, D_6\},\\
&[D_4, D_6] = -\{D_0, D_4\} + \frac{35}{2}D_0,\ [D_4, D_7] = \frac{1}{2}\{D_1 - D_2, D_4\}\\
&\qquad + \frac{35}{4}(D_2 - D_1),\ [D_4, D_8] = \frac{1}{2}\{D_1 - D_2, D_6\} - \{D_0, D_7\},\\
&[D_4, D_9] = -9\{D_0, D_6\},\ [D_5, D_6] = \{D_0, D_5\} - \frac{35}{2}D_0,\\
&[D_5, D_7] = \{D_3, D_6\} + \{D_0, D_8\}, [D_5, D_8] = \{D_3, D_5\} - \frac{35}{2}D_3,\\
&[D_5, D_9] = 9\{D_0, D_6\},\ [D_6, D_7] = \frac{1}{4}\{D_1 - D_2, D_6\} + \frac{1}{2}\{D_3, D_4\}\\
&\qquad + \frac{1}{2}\{D_0, D_7\} - \frac{35}{4}D_3,\ [D_6, D_8] = \frac{1}{4}\{D_1 - D_2, D_5\} + \frac{1}{2}\{D_3, D_6\}\\
&\qquad - \frac{1}{2}\{D_0, D_8\} + \frac{35}{8}(D_2 - D_1),\ [D_6, D_9] = \frac{9}{2}\{D_0, D_4 - D_5\},\\
&[D_7, D_8] = -\frac{1}{4}\{D_0, \{D_1, D_2\}\} + \frac{1}{2}\{D_0, D_3^2\} + \frac{1}{2}\{D_0, D_9\} + \frac{1}{4}\{D_1 - D_2, D_8\}\\
&\qquad + \frac{1}{4}\{D_0, D_5\} + \frac{283}{64}\{D_0, D_1 + D_2\} - \frac{175}{2}D_0 - \frac{1}{2}\{D_3, D_7\} + 5D_0^3\\
&\qquad + \frac{1}{4}\{D_0, D_4\},\ [D_7, D_9] = \frac{1}{4}\{\{D_0, D_7\}, D_6\} + \frac{1}{8}\{D_2 - D_1, \{D_4, D_5\}\}\\
&\qquad - \frac{1}{4}\{\{D_0, D_4\}, D_8\} + \frac{1}{4}\{D_1 - D_2, D_6^2\} - \frac{1}{2}\{D_0, D_8\} + \frac{25}{32}\{D_3, D_6\}\\
&\qquad + \frac{185}{64}\{D_1 - D_2, D_4\} + \frac{17}{8}\{D_1 - D_2, D_5\} + \frac{35 \cdot 181}{128}(D_2 - D_1),
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
[D_8, D_9] &= -\frac{1}{4}\{\{D_0, D_6\}, D_8\} - \frac{1}{4}\{D_3, \{D_4, D_5\}\} + \frac{1}{4}\{\{D_0, D_7\}, D_5\} \\
&+ \frac{1}{2}\{D_3, D_6^2\} + \frac{169}{32}\{D_3, D_5\} + \frac{45}{64}\{D_1 - D_2, D_6\} + \frac{37}{8}\{D_3, D_4\} \\
&+ \frac{5}{8}\{D_0, D_7\} - \frac{35 \cdot 177}{64} D_3 \,.
\end{aligned}
$$

One can verify by direct calculations that all elements from $\mathrm{ZDiff}_I(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}})$ of degrees $\leqslant 4$ are linear combinations of elements C_1, C_1^2 and C_2, where

$$
\begin{aligned}
C_1 &= D_0^2 + D_1 + D_2 + D_4 + D_5, \\
C_2 &= \frac{1}{2}\{D_4, D_5\} - D_6^2 - 2D_9 + \frac{189}{16}(D_1 + D_2) + \frac{5}{4}(D_4 + D_5) \,.
\end{aligned}
$$

Using substitution (3.43) one can obtain from relations above the commutator relations for the algebra $\mathrm{Diff}_I(\mathbf{H}^2(\mathbb{C}a)_{\mathbf{S}})$:

$$
\begin{aligned}
&[\bar D_0, \bar D_1] = \bar D_3,\ [\bar D_0, \bar D_2] = \bar D_3,\ [\bar D_0, \bar D_3] = \frac{1}{2}(\bar D_1 + \bar D_2),\ [\bar D_0, \bar D_4] = 2\bar D_6, \\
&[\bar D_0, \bar D_5] = 2\bar D_6,\ [\bar D_0, \bar D_6] = \bar D_4 + \bar D_5, [\bar D_0, \bar D_7] = \bar D_8, [\bar D_0, \bar D_8] = \bar D_7, \\
&[\bar D_0, \bar D_9] = 0,\ [\bar D_1, \bar D_2] = -\{\bar D_0, \bar D_3\} - 2\bar D_7,\ [\bar D_1, \bar D_3] = -\frac{1}{2}\{\bar D_0, \bar D_1\} - \bar D_8 \\
&\qquad - 10\bar D_0,\ [\bar D_1, \bar D_4] = -2\bar D_7, [\bar D_1, \bar D_5] = 0,\ [\bar D_1, \bar D_6] = -\bar D_8, \\
&[\bar D_1, \bar D_7] = -\frac{1}{2}\{\bar D_1, \bar D_2 + \bar D_4\} + \bar D_9 + \frac{1}{2}\{\bar D_3, \bar D_6\} + \bar D_3^2 + 5\bar D_0^2 + \frac{3}{32}\bar D_1 \\
&\qquad - \frac{283}{32}\bar D_2 - \frac{19}{2}\bar D_4 - \frac{1}{2}\bar D_5,\ [\bar D_1, \bar D_8] = \frac{1}{2}\{\bar D_3, \bar D_5\} - \frac{1}{2}\{\bar D_1, \bar D_6\} \\
&\qquad - 10\bar D_6 - \frac{35}{4}\bar D_3,\ [\bar D_1, \bar D_9] = -\frac{1}{2}\{\bar D_5, \bar D_7\} + \frac{1}{2}\{\bar D_6, \bar D_8\} \\
&\qquad + \frac{189}{32}\{\bar D_0, \bar D_3\} + \frac{169}{16}\bar D_7, [\bar D_2, \bar D_3] = \frac{1}{2}\{\bar D_0, \bar D_2\} + \bar D_8 - 10\bar D_0, \\
&[\bar D_2, \bar D_4] = -2\bar D_7,\ [\bar D_2, \bar D_5] = 0, [\bar D_2, \bar D_6] = -\bar D_8,\ [\bar D_2, \bar D_7] = -\frac{1}{2}\{\bar D_2, \bar D_1 - \bar D_4\} \\
&\qquad + \bar D_9 - \frac{1}{2}\{\bar D_3, \bar D_6\} + \bar D_3^2 + 5\bar D_0^2 - \frac{3}{32}\bar D_2 + \frac{283}{32}\bar D_1 - \frac{19}{2}\bar D_4 - \frac{1}{2}\bar D_5, \\
&[\bar D_2, \bar D_8] = \frac{1}{2}\{\bar D_2, \bar D_6\} - \frac{1}{2}\{\bar D_3, \bar D_5\} + \frac{35}{4}\bar D_3 - 10\bar D_6, \\
&[\bar D_2, \bar D_9] = -\frac{1}{2}\{\bar D_5, \bar D_7\} + \frac{1}{2}\{\bar D_6, \bar D_8\} + \frac{189}{32}\{\bar D_0, \bar D_3\} + \frac{169}{16}\bar D_7, [\bar D_3, \bar D_4] = 0, \\
&[\bar D_3, \bar D_5] = 2\bar D_8,\ [\bar D_3, \bar D_6] = \bar D_7,\ [\bar D_3, \bar D_7] = -\frac{1}{4}\{\bar D_1 - \bar D_2, \bar D_6\} - 10\bar D_6,
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
&[\bar D_3, \bar D_8] = \frac{1}{2}\{\bar D_1, \bar D_2\} - \frac{1}{4}\{\bar D_1 - \bar D_2, \bar D_5\} - \bar D_9 - \bar D_3^2 - 5\bar D_0^2 \\
&\qquad - \frac{143}{32}(\bar D_1 - \bar D_2) - \frac{1}{2}\bar D_4 - \frac{19}{2}\bar D_5,\ [\bar D_3, \bar D_9] = \frac{1}{2}\{\bar D_4, \bar D_8\} \\
&\qquad - \frac{1}{2}\{\bar D_6, \bar D_7\} + \frac{189}{64}\{\bar D_0, \bar D_1 + \bar D_2\} + \frac{169}{16}\bar D_8,
\end{aligned}
$$

$$
\begin{aligned}
&[\bar D_4, \bar D_5] = -2\{\bar D_0, \bar D_6\},\ [\bar D_4, \bar D_6] = -\{\bar D_0, \bar D_4\} - \frac{35}{2}\bar D_0,\\
&[\bar D_4, \bar D_7] = \frac{1}{2}\{\bar D_1 + \bar D_2, \bar D_4\} + \frac{35}{4}(\bar D_2 + \bar D_1),\\
&[\bar D_4, \bar D_8] = \frac{1}{2}\{\bar D_1 + \bar D_2, \bar D_6\} - \{\bar D_0, \bar D_7\},\\
&[\bar D_4, \bar D_9] = 9\{\bar D_0, \bar D_6\},\ [\bar D_5, \bar D_6] = \{\bar D_0, \bar D_5\} - \frac{35}{2}\bar D_0,\ [\bar D_5, \bar D_7] = \{\bar D_3, \bar D_6\}\\
&\qquad + \{\bar D_0, \bar D_8\}, [\bar D_5, \bar D_8] = \{\bar D_3, \bar D_5\} - \frac{35}{2}\bar D_3,\ [\bar D_5, \bar D_9] = 9\{\bar D_0, \bar D_6\},\\
&[\bar D_6, \bar D_7] = \frac{1}{4}\{\bar D_1 + \bar D_2, \bar D_6\} + \frac{1}{2}\{\bar D_3, \bar D_4\}\\
&\qquad + \frac{1}{2}\{\bar D_0, \bar D_7\} + \frac{35}{4}\bar D_3,\ [\bar D_6, \bar D_8] = \frac{1}{4}\{\bar D_1 + \bar D_2, \bar D_5\} + \frac{1}{2}\{\bar D_3, \bar D_6\}\\
&\qquad - \frac{1}{2}\{\bar D_0, \bar D_8\} - \frac{35}{8}(\bar D_2 + \bar D_1),\ [\bar D_6, \bar D_9] = \frac{9}{2}\{\bar D_0, \bar D_4 + \bar D_5\},\\
&[\bar D_7, \bar D_8] = -\frac{1}{4}\{\bar D_0, \{\bar D_1, \bar D_2\}\} + \frac{1}{2}\{\bar D_0, \bar D_3^2\} + \frac{1}{2}\{\bar D_0, \bar D_9\} + \frac{1}{4}\{\bar D_1 + \bar D_2, \bar D_8\}\\
&\qquad - \frac{1}{4}\{\bar D_0, \bar D_5\} + \frac{283}{64}\{\bar D_0, \bar D_1 - \bar D_2\} + \frac{175}{2}\bar D_0 - \frac{1}{2}\{\bar D_3, \bar D_7\} + 5\bar D_0^3\\
&\qquad + \frac{1}{4}\{\bar D_0, \bar D_4\},\ [\bar D_7, \bar D_9] = \frac{1}{4}\{\{\bar D_0, \bar D_7\}, \bar D_6\} - \frac{1}{8}\{\bar D_2 + \bar D_1, \{\bar D_4, \bar D_5\}\}\\
&\qquad - \frac{1}{4}\{\{\bar D_0, \bar D_4\}, \bar D_8\} + \frac{1}{4}\{\bar D_1 + \bar D_2, \bar D_6^2\} + \frac{1}{2}\{\bar D_0, \bar D_8\}\\
&\qquad - \frac{25}{32}\{\bar D_3, \bar D_6\} + \frac{185}{64}\{\bar D_1 + \bar D_2, \bar D_4\} - \frac{17}{8}\{\bar D_1 + \bar D_2, \bar D_5\}\\
&\qquad + \frac{35\cdot 181}{128}(\bar D_2 + \bar D_1),\ [\bar D_8, \bar D_9] = -\frac{1}{4}\{\{\bar D_0, \bar D_6\}, \bar D_8\}\\
&\qquad - \frac{1}{4}\{\bar D_3, \{\bar D_4, \bar D_5\}\} + \frac{1}{4}\{\{\bar D_0, \bar D_7\}, \bar D_5\}\\
&\qquad + \frac{1}{2}\{\bar D_3, \bar D_6^2\} - \frac{169}{32}\{\bar D_3, \bar D_5\} + \frac{45}{64}\{\bar D_1 + \bar D_2, \bar D_6\}\\
&\qquad + \frac{37}{8}\{\bar D_3, \bar D_4\} + \frac{5}{8}\{\bar D_0, \bar D_7\} + \frac{35\cdot 177}{64}\bar D_3\ .
\end{aligned}
$$

The analogs in $\mathrm{ZDiff}_I(\mathbf{H}^2(\mathbb{C}a)_\mathbf{S})$ of operators C_1 and C_2 are

$$
\begin{aligned}
\overline{C}_1 &= \bar D_0^2 + \bar D_1 - \bar D_2 + \bar D_4 - \bar D_5,\\
\overline{C}_2 &= \frac{1}{2}\{\bar D_4, \bar D_5\} - \bar D_6^2 - 2\bar D_9 + \frac{189}{16}(\bar D_1 - \bar D_2) + \frac{5}{4}(\bar D_4 - \bar D_5)\ .
\end{aligned}
$$

3.6 The Kernel of the Operator D_0

The theorem below concerns the kernel of the operator D_0, constructed above for all two-point homogeneous Riemannian spaces.

Theorem 3.1 ([168]). *Let Q be a two-point homogeneous Riemannian space and G be the identity component of the isometry group for Q. For every smooth* *vector field v on Q define a function f_v on $Q_{\mathbf{S}}$ by the following formula:*

$$f_v(y) = \hat{g}(v(x), \xi) \equiv \langle v(x), \xi \rangle,$$

where $x \in Q$, $\hat{g}(\cdot,\cdot) \equiv \langle \cdot,\cdot \rangle$ is the Riemannian metric on Q, $\xi \in T_xQ$, $\langle \xi, \xi \rangle = 1$, $y = (x, \xi) \in Q_{\mathbf{S}}$. Let $D_0 \in \operatorname{Diff}_G(Q_{\mathbf{S}})$ be the differential operator constructed in section 3.1 (for the noncompact case see proposition 1.5). For every element $X \in \mathfrak{g}$ denote by $\widetilde{X}$ the corresponding Killing vector field on Q. Then the identity $D_0 f_v \equiv 0$ on $Q_{\mathbf{S}}$ is equivalent to the equality $v = \widetilde{X}$ for some $X \in \mathfrak{g}$.

This theorem for the case $Q = \mathbf{H}^n(\mathbb{R})$ was formulated and proved in [146] by the explicit coordinate calculations. The proof below valid in the general case is more conceptual.

Proof. Choose an arbitrary point $x_0 \in Q$ and let $e_0 = \frac{1}{R}\tilde{\Lambda}(x_0) \in T_{x_0}Q$, where Λ and R are from Proposition 1.4. Then $\langle e_0, e_0 \rangle = 1$. The space $Q_{\mathbf{S}}$ is the G-orbit Gy_0, where $y_0 = (x_0, e_0) \in Q_{\mathbf{S}}$.

The action of D_0 on f_v can be written in the following way (see (2.7)):

$$(D_0 f_v)(gy_0) = \left.\frac{d}{dt}\right|_{t=0} f_v\left(g \exp(t\Lambda) y_0\right),\ g \in G.$$

Therefore,

$$\begin{aligned}
(D_0 f_v)(gy_0) &= \left.\frac{d}{dt}\right|_{t=0} \langle v\left(g\exp(t\Lambda)x_0\right), g\exp(t\Lambda)e_0\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \left\langle v\left(g\exp(t\Lambda)g^{-1}gx_0\right), g\exp(t\Lambda) \left.\frac{d}{d\mu}\right|_{\mu=0} \exp(\mu\Lambda)x_0 \right\rangle \\
= \left.\frac{d}{dt}\right|_{t=0} &\left\langle v\left(\exp(t\operatorname{Ad}_g\Lambda)gx_0\right), \left.\frac{d}{d\mu}\right|_{\mu=0} \exp(t\operatorname{Ad}_g\Lambda)\exp(\mu\operatorname{Ad}_g\Lambda)gx_0 \right\rangle \\
= \left.\frac{d}{dt}\right|_{t=0} &\left\langle v\left(\exp(t\operatorname{Ad}_g\Lambda)gx_0\right), \left.\frac{d}{d\mu}\right|_{\mu=0} \exp(\mu\operatorname{Ad}_g\Lambda)\exp(t\operatorname{Ad}_g\Lambda)gx_0 \right\rangle .
\end{aligned}$$

Due to the transitivity of G-action on $Q_{\mathbf{S}}$ the point $y := (x, e) := (gx_0, \widetilde{\operatorname{Ad}_g\Lambda}|_{gx_0})$ can be considered as an arbitrary one. Denote $W = \operatorname{Ad}_g\Lambda$. Then one gets

$$(D_0 f_v)(y) = \left.\frac{d}{dt}\right|_{t=0} \left\langle v\left(\exp(tW)x\right), \widetilde{W}\left(\exp(tW)x\right)\right\rangle = \pounds_{\widetilde{W}}\hat{g}(v(x), \widetilde{W}(x)),$$

where $\pounds_X$ is the Lie derivative along the vector field X. The vector field $\widetilde{W}$ is Killing, so $\pounds_{\widetilde{W}}\hat{g} = 0$ and

$$\begin{aligned}
D_0 f_v &= \hat{g}(\pounds_{\widetilde{W}}v, \widetilde{W}) + \hat{g}(v, \pounds_{\widetilde{W}}\widetilde{W}) = \hat{g}(\pounds_{\widetilde{W}}v, \widetilde{W}) = -\hat{g}(\pounds_v\widetilde{W}, \widetilde{W}) \\
&= \frac{1}{2}\left(\pounds_v\hat{g}\right)(\widetilde{W}, \widetilde{W}) - \frac{1}{2}\pounds_v\left(\hat{g}(\widetilde{W}, \widetilde{W})\right) = \frac{1}{2}\left(\pounds_v\hat{g}\right)(\widetilde{W}, \widetilde{W}),
\end{aligned} \tag{3.46}$$

due to $\hat{g}(\widetilde{W}(x), \widetilde{W}(x)) = \hat{g}(g\tilde{\Lambda}(x_0), g\tilde{\Lambda}(x_0)) = \hat{g}(\tilde{\Lambda}(x_0), \tilde{\Lambda}(x_0)) = R^2$ and $\pounds_X Y = [X, Y]$. The element $\widetilde{W} \in T_x Q$ is arbitrary, therefore from (3.46) one sees that the condition $D_0 f_v = 0$ is equivalent to the equality $\pounds_v \hat{g} = 0$. This means that v is a Killing vector field and has the form $v = \widetilde{X}$ for some $X \in \mathfrak{g}$ if and only if $D_0 f_v \equiv 0$. □

4

Hamiltonian Systems with Symmetry and Their Reduction

In this chapter we shall give a brief description of classical Hamiltonian mechanics with symmetry, emphasizing the mechanics on cotangent bundles, the Hamiltonian reduction, the commutative and noncommutative integrability and the transition from quantum mechanics to classical one. Another purpose of this chapter is to fix some notations. Until indicated otherwise, the material of this chapter is quite standart and different parts of it could be found in [8, 32, 58, 193, 200] and many other sources. It will be used below in Chaps. 6 and 7 for studying the classical one- and two-body problems on two-point homogeneous Riemannian spaces.

4.1 Basic Facts from Hamiltonian Mechanics

A vector field X on a smooth manifold M is called *complete* [92] if the corresponding local one-parameter group of diffeomorphisms of M is global. Recall that a smooth manifold M endowed with a closed nondegenerate 2-form ω is called *symplectic* and the form ω is called a *symplectic structure*. A symplectic manifold is necessarily even-dimensional. An every smooth function h on M defines the *Hamiltonian vector field* X_h on M such that

$$dh = \omega(\cdot, X_h) \equiv -i_{X_h}\omega, \tag{4.1}$$

where $i_X\omega$ is the contraction of the vector field X and the form ω. The function h is conserved by the flow, generated by the field X_h, and is called the *Hamiltonian function* with respect to this *Hamiltonian flow* and the corresponding *Hamiltonian system of differential equations* of the first order. Thus, the *Hamiltonian system* is the triple (M, ω, h). In mechanics the manifold M is called the *phase space*.

The form ω is conserved by any Hamiltonian flow. Indeed,

$$\pounds_{X_h}\omega = d \circ i_{X_h}\omega + i_{X_h} \circ d\omega = -d \circ dh + i_{X_h}0 = 0\ , \tag{4.2}$$

where we used the Cartan formula [32, 143]:

$$\pounds_X\omega' = (d \circ i_X + i_X \circ d)\omega', \tag{4.3}$$

valid for every vector field X and every differential form ω'.

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For two smooth functions f and g on M their *Poisson brackets* are

$$[f,h]_P := -\omega(X_f, X_h) = -dh(X_f) = df(X_h) \ . \tag{4.4}$$

Obviously, it holds $[f,g]_P = -[g,f]_P$ and $[f,f]_P = 0$. Functions f and h are *in involution* if $[f,g]_P = 0$ or, equivalently, if f is an integral for the Hamiltonian vector field X_h, i.e., if f is constant along trajectories of the flow, generated by X_h. In this case h is also an integral for the Hamiltonian vector field X_f.

Proposition 4.1. *The vector field $[X_f, X_h]$ is Hamiltonian with the Hamiltonian function $-[f,h]_P$. Poisson brackets satisfy the Jacobi identity*

$$[f,[g,h]_P]_P + [h,[f,g]_P]_P + [g,[h,f]_P]_P = 0 \tag{4.5}$$

and the Leibniz rule

$$[f, gh]_P = [f,g]_P h + g[f,h]_P \ , \tag{4.6}$$

where $f, g, h \in C^\infty(M)$.

Proof. The identity

$$i_{[X,Y]} = [\pounds_X, i_Y]$$

for arbitrary smooth vector fields X and Y is well known in calculus on manifolds [32, 143]. Therefore, using (4.2) and the Cartan formula (4.3) once again, one gets:

$$\begin{aligned} i_{[X_f,X_h]}\omega &= \pounds_{X_f} \circ i_{X_h}\omega - i_{X_h} \circ \pounds_{X_f}\omega = d \circ i_{X_f} \circ i_{X_h}\omega + i_{X_f} \circ d \circ i_{X_h}\omega \\ &= d\left(\omega(X_h, X_f)\right) - i_{X_f} \circ d \circ dh = d[f,h]_P \end{aligned}$$

that proves the first claim of the proposition. Further from (4.4) and the first claim it follows:

$$\begin{aligned} [f,[g,h]_P]_P + [g,[h,f]_P]_P &= X_f \circ X_g h - X_g \circ X_f h \\ &= [X_f, X_g]h = -X_{[f,g]_P} h = -[h,[f,g]_P]_P \end{aligned}$$

that proves the second claim. At last

$$[f, gh]_P = -X_f(gh) = -X_f(g)h - gX_f(h) = [f,g]_P h + g[f,h]_P \ .$$

□

Definition 4.1. *A Poisson algebra $\mathcal{A}$ is a commutative associative algebra endowed with a bilinear skew-symmetric operation satisfying the Jacobi identity and the Leibniz rule.*

Thus, we see that the algebra $C^\infty(M)$ is Poisson with respect to the brackets $[\cdot,\cdot]_P$. Neglecting pointwise multiplication in $C^\infty(M)$ one obtains an infinite dimensional Lie algebra and the map $h \to X_h$, $h \in C^\infty(M)$ is an antihomomorphism onto the Lie algebra of Hamiltonian vector fields on M.

Definition 4.2. *A Hamiltonian system* (M, ω, h) *is called completely integrable (in commutative sense) if there are* $n = \frac{1}{2}\dim M$ *smooth functions* $f_1 = h, f_2, \ldots, f_n$ *on* M *such that*

1. *they are in the involution:* $[f_i, f_j]_P = 0$*;*
2. *the differential forms* df_i, $i = 1, \ldots, n$ *are linearly independent at every point of some open dense subset* $M' \subset M$.

Theorem 4.1 (Arnold-Liouville, [8, 32]). *Let* (M, ω, h) *be a completely integrable Hamiltonian system with integrals* $f_1 = h, f_2, \ldots, f_n$*. Let also* $c \in \mathbb{R}^n$ *be a regular value of the map* $\mathbf{f} := (f_1, \ldots, f_n) : M \to \mathbb{R}^n$.[1] *Then*

1. *the corresponding level set* $M_c = f^{-1}(c)$ *is a smooth Lagrangian submanifold of* M*, i.e., it is* n*-dimensional and* $\omega|_{M_c} = 0$*;*
2. *any compact connected component of* M_c *is diffeomorphic to the torus* $\mathbb{T}^n$ *and the vector fields* $X_{f_1}, \ldots, X_{f_1}$ *on it are complete;*
3. *if the Hamiltonian flows of the vector fields* $X_{f_1}, \ldots, X_{f_1}$ *starting at a point* $x \in M_c$ *are complete, then the connected component of* M_c *containing* x *is a homogeneous space for the additive group* $\mathbb{R}^n$ *and the flows of the vector fields* $X_{f_1}, \ldots, X_{f_1}$ *are linear w.r.t. corresponding coordinates* $\varphi_1, \ldots, \varphi_n$*, known as angle coordinates.*

The concept of the *noncommutative integrability* was firstly introduced in [123] for Lie algebras of integrals. For nonlinear algebras it was later generalized in [29]. This concept is more general than Definition 4.2. Firstly, we shall formulate the analytic version of this concept, which is preliminary for the geometric one. Here we follow [23] and [24].

Let $\mathcal{F}$ be a subalgebra of the Poisson algebra $C^\infty(M)$, functionally generated by functions $f_1, \ldots, f_l$. Suppose that 1-forms $df_1, \ldots, df_l$ are linearly independent at every point of an open dense subset $M' \subset M$ and $\operatorname{rank} \|[f_i, f_j]_P\| =: l - r = \text{const}$ everywhere in M'.

Theorem 4.2 ([23, 24]). *Let* $l + r = \dim M = 2n$ *and let* $c \in \mathbb{R}^l$ *be a regular value of the map* $\mathbf{f} : M \to (f_1, \ldots, f_l) \in \mathbb{R}^l$*, existing due to the independence of* $df_1, \ldots, df_l$ *on* M'*. Suppose that a Hamiltonian function* $h \in C^\infty(M)$ *commutes with the algebra* $\mathcal{F}$*. Then*

1. $M_c = \mathbf{f}^{-1}(c)$ *is an isotropic (i.e.,* $\omega|_{M_c} = 0$*)* r*-dimensional submanifold of* M,
2. *the Hamiltonian system with the Hamiltonian function* h *is completely integrable in the sense of Definition 4.2,*
3. *a compact connected component* T_c^r *of* M_c *is diffeomorphic to a* r*-dimensional torus. In some neighborhood* U *of* T_c^r *there are generalized action-angle variables* y_i, x_i, I_i *and* $\varphi_i \mod 2\pi$ *such that*

$$\omega|_U = \sum_{i=1}^{r} dI_i \wedge d\varphi_i + \sum_{i=1}^{n-r} dy_i \wedge dx_i$$

[1] Contrary to the agreement in the Sard theorem [143] we call $c \in \mathbb{R}^n$ a *regular value* of the map $\mathbf{f}$ iff the set $\mathbf{f}^{-1}(c)$ *is nonempty* and $\operatorname{im} d_x\mathbf{f} = \mathbb{R}^n$, $\forall x \in \mathbf{f}^{-1}(c)$, since the empty set is not interesting in mechanics as a phase space.

and the function h depends only on variables I_i, $i = 1, \dots, r$. *The invariant tori in* U *corresponding to other regular values of* $\mathbf{f}$ *are given as the level sets of integrals* I_i, y_j, x_k. *The Hamiltonian equations for the function* h *on* T_c^r *are*

$$\dot{\varphi}_i = \frac{\partial h}{\partial I_i},\ i = 1, \dots, r\ .$$

Under the assumptions of this theorem the Hamiltonian system corresponding to the function h is called *completely integrable in noncommutative sense*. In the case $r = l = n$ this theorem is equivalent to Theorem 4.1.

Below we shall use the following theorem about a map of a constant rank. This theorem appears in different forms. The first form can be found in [40] (10.3.1) for linear spaces, but due to its local character, it is valid also for manifolds.

Theorem 4.3. *Let* N_1 *and* N_2 *be smooth manifolds and* $f : N_1 \to N_2$ *be a smooth map such that in some neighborhood* U *of a point* $x_0 \in N_1$ *the rank of* f *is constant:*

$$\operatorname{rk} f|_U := \operatorname{rk} d_x f|_{x \in U} = k = \text{const}\ .$$

Then there are local coordinates $x_1, \dots, x_n$ *in some neighborhood* $U' \subset U$ *and* $y_1, \dots, y_m$ *in some neighborhood* $V' \subset N_2$ *of the point* $f(x_0)$ *such that* $f(U') \subset V'$ *and the map* $f|_{U'}$ *acts by the formula:*

$$f(x_1, \dots, x_n) = (y_1, \dots, y_m) = (x_1, \dots, x_k, 0, \dots, 0)\ .$$

Another version of this result is as follows.

Theorem 4.4. *Let* $f_1, \dots, f_m$ *be smooth functions on a smooth manifold* N *such that*

$$\dim \operatorname{span}(df_i,\ i = 1, \dots, m) = \dim \operatorname{span}(df_i,\ i = 1, \dots, k) = k,\ k < m$$

in an open set $U \subset N$. *Then for some* C^1 *functions* F_i *on* $\mathbb{R}^k$ *it holds*

$$f_i = F_i(f_1, \dots, f_k),\ i = k + 1, \dots, m$$

in some open set $U' \subset U$.

For N being a linear space this theorem can be found in [157] (III,9;33) and [172] (Sect. 1.74), but again due to its local character, it is valid also for N being a smooth manifold.

Definition 4.3. *Let* $\mathcal{F}$ *be a subalgebra of the Poisson algebra* $C^\infty(M)$ *and* $\operatorname{reg} \mathcal{F}$ *be an open dense subset in* M *such that*

1. $\operatorname{ddim} \mathcal{F} := \dim \operatorname{span}(d_x f,\ f \in \mathcal{F}) = \text{const},\ x \in \operatorname{reg} \mathcal{F}$;
2. *the kernel of the Poisson structure* $[\cdot,\cdot]_P$ *restricted onto* $\operatorname{span}(d_x f,\ f \in \mathcal{F})$ *has a constant dimension* $\operatorname{dind} \mathcal{F}$ *on* $\operatorname{reg} \mathcal{F}$.

The numbers $\operatorname{ddim} \mathcal{F}$ *and* $\operatorname{dind} \mathcal{F}$ *are called respectively differential dimension and differential index of* $\mathcal{F}$. *The algebra* $\mathcal{F}$ *is called complete if*

$$\operatorname{ddim} \mathcal{F} + \operatorname{dind} \mathcal{F} = \dim M\ .$$

Remark 4.1. *For the algebra $\mathcal{F}$ from Definition 4.3 denote by $W_{\mathcal{F}}(x)$ the linear subspace of T_xM, consisting of elements $X_f|_x$, $f \in \mathcal{F}$. Obviously, it holds*

$$\operatorname{ddim} \mathcal{F} = \dim W_{\mathcal{F}}(x), \ \operatorname{dind} \mathcal{F} = \dim \left(W_{\mathcal{F}}(x) \cap W_{\mathcal{F}}(x)^{\omega}\right), \ x \in \operatorname{reg} \mathcal{F},$$

where $W_{\mathcal{F}}(x)^{\omega}$ is the subspace in T_xM, skew-orthogonal to $W_{\mathcal{F}}(x)$ w.r.t. the symplectic form ω. Since

$$\operatorname{dind} \mathcal{F} \leqslant \dim W_{\mathcal{F}}(x)^{\omega} = \dim M - \dim W_{\mathcal{F}}(x),$$

one has

$$\operatorname{ddim} \mathcal{F} + \operatorname{dind} \mathcal{F} \leqslant \dim M$$

and the algebra $\mathcal{F}$ is complete iff $W_{\mathcal{F}}(x)^{\omega} \subset W_{\mathcal{F}}(x)$, $x \in \operatorname{reg} \mathcal{F}$, in other words iff spaces $W_{\mathcal{F}}(x)$, $x \in \operatorname{reg} \mathcal{F}$ are coisotropic.

The existence of a complete algebra of integrals is connected with the noncommutative integrability. Indeed, let $h \in C^{\infty}(M)$ and $[h, \mathcal{F}]_P = 0$ for a complete algebra $\mathcal{F} \subset C^{\infty}(M)$. Let $x_0 \in \operatorname{reg} \mathcal{F}$. There are a neighborhood U of x_0 and functions $f_1, \ldots, f_k \in \mathcal{F}$, $k = \operatorname{ddim} \mathcal{F}$ such that $\dim \operatorname{span}(df_i,\ i = 1, \ldots, k) = k$ on U. According to Theorem 4.4 there is some neighborhood $U' \subset U$ of x_0 such that

$$[f_i, f_j]_P = a_{ij}(f_1, \ldots, f_k) \tag{4.7}$$

on U' for smooth functions a_{ij} on $\mathbb{R}^k$. Suppose that the Hamiltonian flow ϕ_t^h, corresponding to the function h, is a global one at least for points from U'. Let

$$V := \bigcup_{t \in \mathbb{R}} \phi_t^h(U') \ .$$

Since the differential form $df_1 \wedge \ldots \wedge df_k$ and relations (4.7) are invariant w.r.t. the flow ϕ_t^h, we obtain that functions $f_1, \ldots, f_k$ are independent on V and satisfy (4.7) on V. Therefore, the restriction of the Hamiltonian function h onto V obeys the conditions of Theorem 4.2. This motivates the following final definition of complete integrability in noncommutative sense.

Definition 4.4. *A Hamiltonian system is completely integrable in the noncommutative sense if it admits a complete algebra of integrals.*

In Sect. 6.1 we shall prove the noncommutative integrability of the one-particle motion in a central field on two-point homogeneous spaces.

4.2 Hamiltonian Mechanics with Symmetry

4.2.1 The Poisson Structure on the Algebra $S(\mathfrak{g})$

The presence of a continuous symmetry for Hamiltonian systems leads to the existence of some integrals of motion, sometimes sufficient for commutative or noncommutative integrability.

Evidently, for any Lie algebra $\mathfrak{g}$ the commutator $[\cdot,\cdot]$ in the universal enveloping algebra $U(\mathfrak{g})$ obeys the Jacobi identity and the Leibniz rule. Let $U_1(\mathfrak{g}) \subset U_2(\mathfrak{g}) \subset \ldots \subset U(\mathfrak{g})$ be the standard filtration of the algebra $U(\mathfrak{g})$ as in Sect. 2.1.2 and $u \in U_k(\mathfrak{g}), v \in U_l(\mathfrak{g})$. Then for elements $u + U_{k-1}(\mathfrak{g}), v + U_{l-1}(\mathfrak{g}) \in \operatorname{gr} U(\mathfrak{g})$ put

$$[u + U_{k-1}(\mathfrak{g}), v + U_{l-1}(\mathfrak{g})]_P := [u,v] + U_{k+l-2}(\mathfrak{g}).$$

This equality is well defined since

$$[u, U_{l-1}(\mathfrak{g})] \subset U_{k+l-2}(\mathfrak{g}),\ [U_{k-1}(\mathfrak{g}), v] \subset U_{k+l-2}(\mathfrak{g}) \ .$$

Thus, one gets the Poisson brackets $[\cdot,\cdot]_P$ for $\operatorname{gr} U(\mathfrak{g})$. Due to Theorem 2.1, the algebras $\operatorname{gr} U(\mathfrak{g})$ and $S(\mathfrak{g})$ are isomorphic and we obtain the Poisson brackets on $S(\mathfrak{g})$. Obviously, for $S_1(\mathfrak{g}) = \mathfrak{g}$ these brackets coincide with the Lie operation $[\cdot,\cdot]$ and can be uniquely extended onto the whole $S(\mathfrak{g})$ by the linearity and the Leibniz rule. Note that for $u \in S_k(\mathfrak{g}), v \in S_l(\mathfrak{g})$ it holds $[u,v]_P \in S_{k+l-1}(\mathfrak{g})$.

To distinguish the commutative algebra $S(\mathfrak{g})$ and the same algebra endowed with the Poisson structure constructed above, we denote it in the latter case by $S_p(\mathfrak{g})$.

The algebra $S(\mathfrak{g})$ is naturally identified with the algebra $\mathcal{P}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$, which is therefore Poisson. The Poisson structure is extended from $\mathcal{P}(\mathfrak{g}^*)$ onto the algebra $C^\infty(\mathfrak{g}^*)$ by the formula

$$[f,h]_P = \sum_{l,k=1}^{\dim \mathfrak{g}} \frac{\partial f}{\partial x_k}\frac{\partial h}{\partial x_l}[x_k, x_l]_P = \sum_{l,k,i=1}^{\dim \mathfrak{g}} \frac{\partial f}{\partial x_k}\frac{\partial h}{\partial x_l} c^i_{kl} x_i, \tag{4.8}$$

where $f, h \in C^\infty(\mathfrak{g}^*)$; x_k are coordinates on $\mathfrak{g}^*$, corresponding to a basis e_k in $\mathfrak{g}$ and $[e_k, e_l] = c^i_{kl} e_i$.

In fact, the Poisson algebra $S_p(\mathfrak{g})$ is the classical analogue for the algebra $U(\mathfrak{g})$ and the brackets $[\cdot,\cdot]_P$ inherit leading terms properties of the brackets $[\cdot,\cdot]$ in $U(\mathfrak{g})$. The same arguments as in the proof of Proposition 2.2 give:

Proposition 4.2. *Let G be a connected Lie group. The center $\mathrm{Z}\, S_p(\mathfrak{g})$ of the Poisson algebra $S_p(\mathfrak{g})$ (w.r.t. Poisson brackets) consists of Ad_G-invariant elements. Equivalently, the center $\mathrm{Z}\,\mathcal{P}(\mathfrak{g}^*)$ of the Poisson algebra $\mathcal{P}(\mathfrak{g}^*)$ consists of Ad^*_G-invariants.*

Define the *coadjoint action* of the group G on the space $\mathfrak{g}^*$ by the formula

$$\left(\mathrm{Ad}^*_g f\right)(x) = f\left(\mathrm{Ad}_{g^{-1}} x\right),\ g \in G, x \in \mathfrak{g}, f \in \mathfrak{g}^* \ . \tag{4.9}$$

Note that in order to obtain the left Ad^*_G-action we denote here by Ad^*_g the operator adjoint to $\mathrm{Ad}_{g^{-1}}$. Denote by Y_c the vector field on $\mathfrak{g}^*$, corresponding to an element $Y \in \mathfrak{g}$ with respect to Ad^*_G-action:

$$Y_c|_f = \frac{d}{dt}\left(\mathrm{Ad}^*_{\exp(tY)} f\right)\bigg|_{t=0},\ f \in \mathfrak{g}^*,\ \text{ i.e., }\ Y_c|_f(X) = -f([Y,X]),\ X \in \mathfrak{g} \ . \tag{4.10}$$

Obviously, vectors $Y_c|_f$, $Y \in \mathfrak{g}$ span the tangent space $T_f\mathcal{O}$ for an Ad^*_G- orbit $\mathcal{O}$, passing through $f \in \mathfrak{g}^*$.[2]

On the other hand, an element $Y \in \mathfrak{g}$ defines the linear function $Y^{\#}$ on $\mathfrak{g}^* :\ Y^{\#}(f) := f(Y),\ f \in \mathfrak{g}^*$. Motivated by (4.4), one can define the *Kirillov 2-form* ω^* on $\mathcal{O}$:

$$\omega^*(Y^*, Z^*) := -[Y^{\#}, Z^{\#}]_P := -[Y,Z]^{\#},\ Y, Z \in \mathfrak{g}\ . \tag{4.11}$$

The form ω^* is well defined, since $Y_c|_f = 0$ iff $[Y, \mathfrak{g}] \subset \ker f$. Similarly, if $\omega^*(Y_c, Z_c)|_f = 0$, $\forall Z \in \mathfrak{g}$, then $[Y, \mathfrak{g}] \subset \ker f$ and $Y_c|_f = 0$. Hence the form ω^* is nondegenerate.

Using the well-known formula for the exterior derivative of a differential form (see for example [92, 143]) and (2.1), one gets

$$\begin{aligned} d\omega^*(T_c, Y_c, Z_c) &= T_c\left(\omega^*(Y_c, Z_c)\right) - Y_c\left(\omega^*(T_c, Z_c)\right) + Z_c\left(\omega^*(T_c, Y_c)\right) \\ &- \omega^*([T_c, Y_c], Z_c) + \omega^*([T_c, Z_c], Y_c) - \omega^*([Y_c, Z_c], T_c) = -T_c\left([Y,Z]^{\#}\right) \\ &- Y_c\left([Z,T]^{\#}\right) - Z_c\left([T,Y]^{\#}\right) - \left([[T,Y],Z] + [[Z,T],Y] + [[Y,Z],T]\right)^{\#}, \\ &\quad \forall T, Y, Z \in \mathfrak{g}\ . \end{aligned}$$

The sum in the last brackets vanishes due to the Jacobi identity in $\mathfrak{g}$. Thus, from (4.10) one gets:

$$\begin{aligned} d\omega^*(T_c, Y_c, Z_c)|_f &= -T_c\left(f([Y,Z])\right) - Y_c\left(f([Z,T])\right) - Z_c\left(f([T,Y])\right) \\ &= f\left([T,[Y,Z]] + [Y,[Z,T]] + [Z,[T,Y]]\right) = f(0) = 0\ . \end{aligned}$$

Hence the form ω^* is closed and it determines the symplectic structure on $\mathcal{O}$. It is easily seen that the Poisson structure on $\mathcal{O}$, constructed according to (4.4), is the restriction of Poisson structure on $C^\infty(\mathfrak{g}^*)$ defined in (4.8). This means that orbits of the Ad^*_G-action in $\mathfrak{g}^*$ are symplectic leaves of the Poisson structure (4.8) (see for example [8, 193]).

Since the center of the Poisson algebra $\mathcal{P}(\mathfrak{g}^*)$ consists of Ad^*_G-invariants, the center of Poisson algebra $C^\infty(\mathfrak{g}^*)$ consists of functions, constant on all Ad^*_G-orbits in $\mathfrak{g}^*$.

4.2.2 The Poisson Action and the Momentum Map

Definition 4.5. *An action of a Lie group G on a symplectic manifold M is called symplectic if it conserves the symplectic structure on M. Let*

$$\left.\widetilde{Y}\right|_x = \left.\frac{d}{dt}\exp(tY)x\right|_{t=0},\ Y \in \mathfrak{g}, x \in M$$

be a vector field on M, corresponding to an element $Y \in \mathfrak{g}$. A symplectic action of a connected Lie group G is called Poisson if there is a homomorphism $\mu^* : \mathfrak{g} \to C^\infty(M)$ of Lie algebras such that $\mu^*(Y)$ is the Hamiltonian function

[2] Due to Proposition 1.6 Ad^*_G-orbits does not depend on the choice of connected Lie group G, corresponding to the fixed algebra $\mathfrak{g}$.

of the vector field $\widetilde{Y}$ *on* M *for every element* $Y \in \mathfrak{g}$*. This homomorphism is called a comomentum map and the map* $\mu : M \to \mathfrak{g}^*$*, defined by the equality* $\mu(x)(Y) = \mu^*(Y)(x)$ *is called a momentum map.*

Example. The Ad_G^*-action on any orbit $\mathcal{O} \subset \mathfrak{g}^*$ is Poisson since the map $\mu^* : \mathfrak{g} \to C^\infty(\mathcal{O}, \mathbb{R})$, defined by the formula $\mu^*(Y) = Y^\#|_{\mathcal{O}}$, $Y \in \mathfrak{g}$, satisfies Definition 4.5 due to (4.10) and (4.11).

Proposition 4.3 ([58, 200]). *If a connected real Lie group* G *is semisimple or at least cohomology groups* $H^1(\mathfrak{g}, \mathbb{R}), H^2(\mathfrak{g}, \mathbb{R})$ *vanish, then every symplectic action of* G *is Poisson.*

Since the Poisson algebra $S_p(\mathfrak{g}) \cong \mathcal{P}(\mathfrak{g}^*)$ is freely generated by elements of $\mathfrak{g}$, the map μ^* is uniquely extended to the homomorphism of Poisson algebras $\mathcal{P}(\mathfrak{g}^*) \to C^\infty(M)$.

There is a natural topology in the algebra $C^\infty(\mathfrak{g}^*)$, generated by seminorms

$$\rho_{\alpha,U}(f) = \sup_U \left| \frac{\partial^\alpha f}{\partial x_\alpha} \right| ,$$

where U is a compact subset of the linear space $\mathfrak{g}^*$, α is a multiindex and x_i, $i = 1, \ldots, \dim \mathfrak{g}$ are affine coordinates in $\mathfrak{g}^*$. The base of neighborhoods of the null function from $C^\infty(\mathfrak{g}^*)$ in this topology consists of sets

$$W_{\alpha,U,\varepsilon} := (f \in C^\infty(\mathfrak{g}^*) | \, \rho_{\alpha,U}(f) < \varepsilon) \ .$$

The algebra $\mathcal{P}(\mathfrak{g}^*)$ is dense in $C^\infty(\mathfrak{g}^*)$ w.r.t. this topology and the homomorphism μ^* is uniquely extended to the continuous homomorphism $C^\infty(\mathfrak{g}^*) \to C^\infty(M)$, for which we save the notation μ^*.

Let M be a symplectic manifold with a Poisson action of a connected Lie group G.

Proposition 4.4. *For every element* $g \in G$ *the following diagram is commutative*

$$\begin{array}{ccc} M & \xrightarrow{\psi_g} & M \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \mathfrak{g}^* & \xrightarrow{\mathrm{Ad}_g^*} & \mathfrak{g}^* \end{array} \tag{4.12}$$

where ψ_g *is the action of* g *on* M*. In other words,* μ *is an equivariant map.*

If $\widetilde{Y}h = 0$ *on* M *for some element* $Y \in \mathfrak{g}$ *and a function* $h \in C^\infty(M)$*, then* $\mu^*(Y)$ *is an integral for the Hamiltonian system, corresponding to* h*.*

Proof. Since a connected Lie group is generated by any neighborhood of its unit [142, 201], it is enough to prove the commutativity of (4.12) for any one parametric subgroup $q(t) = \exp(tY)$, $Y \in \mathfrak{g}, t \in \mathbb{R}$. In other words it is enough to prove the equality

$$\mu(q(t)x)(Z) = \mu(x) \left(\mathrm{Ad}_{q(-t)} Z \right), \forall x \in M, \forall Z \in \mathfrak{g} \ .$$

It is easily seen that the last equality is equivalent to the equality

$$\mu(q(t)x)\left(\mathrm{Ad}_{q(t)} Z\right) = \mu(x)(Z),\ \forall x \in M, \forall Z \in \mathfrak{g}\ . \tag{4.13}$$

From the definitions of μ, μ^* and formula (4.4) one has:

$$\begin{aligned}\frac{d}{dt}\left(\mu(q(t)x)\left(\mathrm{Ad}_{q(t)} Z\right)\right) &= \left(\widetilde{Y}\left(\mu^*\left(\mathrm{Ad}_{q(t)} Z\right)\right)\right)(q(t)x)\\ &\quad + \mu(q(t)x)\left(\left[Y, \mathrm{Ad}_{q(t)} Z\right]\right)\\ &= \left[\mu^*\left(\mathrm{Ad}_{q(t)} Z\right), \mu^*(Y)\right]_P (q(t)x) + \left[\mu^*(Y), \mu^*\left(\mathrm{Ad}_{q(t)} Z\right)\right]_P (q(t)x) = 0\ .\end{aligned}$$

Since $\mu(q(0)x)\left(\mathrm{Ad}_{q(0)} Z\right) = \mu(x)(Z)$, one gets (4.13) that completes the proof of the commutativity of diagram (4.12). The second claim of the proposition is evident since

$$[h, \mu^*(Y)]_P = X_{\mu^*(Y)} h = \widetilde{Y} h = 0$$

due to (4.4) and the definition of μ^*. □

Let G_x be a stationary subgroup of the group G, corresponding to a point $x \in M$ and G_f, $f \in \mathfrak{g}^*$ be a stationary subgroup of G, corresponding to Ad^*_G-action and an element $f \in \mathfrak{g}^*$. Denote by $\mathfrak{g}_x$ and $\mathfrak{g}_f$ the corresponding subalgebras. Due to (4.12) it holds $G_x \subset G_{\mu(x)}$ and $\mathfrak{g}_x \subset \mathfrak{g}_{\mu(x)}$.

One can express the differential of the map μ via symplectic structure. Indeed, from (4.1) for $\xi \in \mathcal{X}(M)$ and $\xi' = \xi|_x$, $x \in M$ one gets:

$$d_x\mu(\xi')(Y) = \xi\left(\mu^*(Y)\right)\big|_x = \omega(\xi, X_{\mu^*(Y)})\big|_x = \omega(\xi', \widetilde{Y})\Big|_x,\ Y \in \mathfrak{g}, \tag{4.14}$$

where d_x is the differential at a point x. Denote by $\tilde{\mathfrak{g}}(x)$ the linear subspace in T_xM spanned by vectors $\widetilde{Y}\Big|_x$ for all elements $Y \in \mathfrak{g}$. From formula (4.14) one obtains that

$$\ker d_x\mu = \tilde{\mathfrak{g}}(x)^\omega, \tag{4.15}$$

where $\tilde{\mathfrak{g}}(x)^\omega$ is the skew-orthogonal complement of $\tilde{\mathfrak{g}}(x)$ in T_xM with respect to the form ω. Since ω is nondegenerate, one has

$$\dim\ker d_x\mu = \dim M - \dim\tilde{\mathfrak{g}}(x)\ .$$

Formula (4.14) implies that $\mathrm{im}\, d_x\mu \subset \mathrm{ann}\,\mathfrak{g}_x \subset \mathfrak{g}^*$. Since $\dim\mathrm{im}\, d_x\mu = \dim M - \dim\ker d_x\mu = \dim\tilde{\mathfrak{g}}(x) = \dim\mathfrak{g} - \dim\mathfrak{g}_x = \dim\mathrm{ann}\,\mathfrak{g}_x$, one gets

$$\mathrm{im}\, d_x\mu = \mathrm{ann}\,\mathfrak{g}_x\ . \tag{4.16}$$

4.2.3 From Momentum Map to Noncommutative Integrability

The momentum map gives a way for constructing complete Poisson algebras.

Let $\mathcal{F}_1 := \mu^*\left(C^\infty(\mathfrak{g}^*)\right)$ and $\mathcal{F}_2$ be a subalgebra of the Poisson algebra $C^\infty(M)$, consisting of G-invariant functions. According to Definition 4.5 at every point $x \in M$ vector fields of the form X_f, $f \in \mathcal{F}_1$ span the space $\tilde{\mathfrak{g}}(x)$. Therefore, it holds $X_f(\mathcal{F}_2) = 0$, $\forall f \in \mathcal{F}_1$ and thus $[\mathcal{F}_1, \mathcal{F}_2]_P = 0$. In other words it holds $\omega\left(W_{\mathcal{F}_1}(x), W_{\mathcal{F}_2}(x)\right) = 0$. Similarly, if $[f, \mathcal{F}_1]_P = 0$ for $f \in C^\infty(M)$, then f is constant along G-orbits in M and $f \in \mathcal{F}_2$.

Summarize the consideration above in the following proposition

Proposition 4.5. *It holds* $\mathrm{Z}\,\mathcal{F}_1 = \mathcal{F}_1 \cap \mathcal{F}_2 \subset \mathrm{Z}\,\mathcal{F}_2$ *and* $W_{\mathcal{F}_1}(x) = \tilde{\mathfrak{g}}(x) \subset W_{\mathcal{F}_2}(x)^\omega, W_{\mathcal{F}_2}(x) \subset W_{\mathcal{F}_1}(x)^\omega = \tilde{\mathfrak{g}}(x)^\omega$ *for every point* $x \in M$.

From here till the end of this section we make the following assumption, concerning G-orbits in M.

Assumption 4.1. *There is an open dense subset* $M' \subset M$ *such that*

1. *it consists of G-orbits in M of a maximal dimension;*
2. $\mathrm{span}(d_x f,\ f \in \mathcal{F}_2) = \mathrm{ann}\,(\tilde{\mathfrak{g}}(x))\,,\ \forall x \in M'$ *or equivalently* $W_{\mathcal{F}_2}(x) = \tilde{\mathfrak{g}}(x)^\omega,\ \forall x \in M'$.

Under this assumption it holds $\dim \mathfrak{g}_x = \mathrm{const}$ for $x \in M'$ and due to (4.16) the restriction of the map μ onto M' is a map of a constant rank. Due to Theorem 4.3 the set $\mu^{-1}(c) \cap M'$ is a smooth manifold for every $c \in \mathfrak{g}^*$.

Obviously, $W_{\mathcal{F}_1}(x) = \tilde{\mathfrak{g}}(x)$ for $x \in M'$ and due to Assumption 4.1 $W_{\mathcal{F}_2}(x) = \tilde{\mathfrak{g}}(x)^\omega = W_{\mathcal{F}_1}(x)^\omega$. Since $(W_{\mathcal{F}_1}(x)^\omega)^\omega = W_{\mathcal{F}_1}(x)$, one obtains also $W_{\mathcal{F}_2}(x)^\omega = W_{\mathcal{F}_1}(x)$, $x \in M'$. Thus, it holds

$$(W_{\mathcal{F}_1}(x) + W_{\mathcal{F}_2}(x))^\omega = W_{\mathcal{F}_1}(x)^\omega \cap W_{\mathcal{F}_2}(x)^\omega = W_{\mathcal{F}_2}(x) \cap W_{\mathcal{F}_1}(x),\ x \in M' \tag{4.17}$$

and

$$W_{\mathcal{F}_1}(x)^\omega \cap W_{\mathcal{F}_1}(x) = W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x) = W_{\mathcal{F}_2}(x)^\omega \cap W_{\mathcal{F}_2}(x),\ x \in M'. \tag{4.18}$$

On the other hand, due to (4.15) one gets

$$W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x) = W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_1}(x)^\omega = \tilde{\mathfrak{g}}(x) \cap \tilde{\mathfrak{g}}(x)^\omega = \tilde{\mathfrak{g}}(x) \cap \ker d_x\mu\,.$$

This means that elements of the space $W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x)$ are in one-to-one correspondence with elements from $\mathfrak{g}_{\mu(x)}$ modulo $\mathfrak{g}_x$, $x \in M'$. One can conclude that

$$\dim\,(W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x)) = \dim\left(\mathfrak{g}_{\mu(x)}/\mathfrak{g}_x\right),\ x \in M'\,.$$

The subset M' from Assumption 4.1 can play roles of $\mathrm{reg}\,\mathcal{F}_1$ and $\mathrm{reg}\,\mathcal{F}_2$. Indeed, for every point $x \in M'$ it holds

$$\mathrm{ddim}\,\mathcal{F}_1 = \dim \mathrm{im}\, d_x\mu = \dim \mathfrak{g}/\mathfrak{g}_x, \tag{4.19}$$

due to (4.16) and

$$\mathrm{ddim}\,\mathcal{F}_2 = \mathrm{codim}\, G\cdot x = \dim M - \dim \mathfrak{g}/\mathfrak{g}_x\,. \tag{4.20}$$

Also, due to (4.18) and Remark 4.1:

$$\mathrm{dind}\,\mathcal{F}_1 = \mathrm{dind}\,\mathcal{F}_2 = \dim\,(W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x)) = \dim\left(\mathfrak{g}_{\mu(x)}/\mathfrak{g}_x\right),\ x \in M'\,. \tag{4.21}$$

Proposition 4.6. *Let* $\mathcal{F} := \mathcal{F}_1 + \mathcal{F}_2$ *be the Poisson algebra generated by* $\mathcal{F}_1$ *and* $\mathcal{F}_2$. *Since* $W_{\mathcal{F}} = W_{\mathcal{F}_1} + W_{\mathcal{F}_2}$, *this algebra is complete with* $\mathrm{reg}\,\mathcal{F} = M'$ *due to remark* 4.1 *and formula* (4.17).

One can see the completeness of $\mathcal{F}$ also by direct calculations:

$$\begin{aligned}\operatorname{ddim}\mathcal{F} + \operatorname{dind}\mathcal{F} &= \dim W_{\mathcal{F}_1}(x) + \dim W_{\mathcal{F}_2}(x) - \dim\left(W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x)\right)\\ &+ \dim\left((W_{\mathcal{F}_1}(x) + W_{\mathcal{F}_2}(x))^{\omega} \cap (W_{\mathcal{F}_1}(x) + W_{\mathcal{F}_2}(x))\right) = \operatorname{ddim}\mathcal{F}_1 + \operatorname{ddim}\mathcal{F}_2\\ &- \dim\left(W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x)\right) + \dim\left(W_{\mathcal{F}_1}(x) \cap W_{\mathcal{F}_2}(x)\right) = \dim M,\ x \in M' ,\end{aligned}$$

due to (4.17), (4.19) and (4.20).

Remark 4.2. *Thus, a G-invariant Hamiltonian function on M, commuting with the algebra $\mathcal{F}$, corresponds to a Hamiltonian system, integrable in noncommutative sense. The calculations above give* $\operatorname{dind}\mathcal{F} = \dim(W_{\mathcal{F}_1} \cap W_{\mathcal{F}_2})$, *$x \in M'$ and a common level set of functions from $\mathcal{F}$ in general position is* $\operatorname{dind}\mathcal{F}$*-dimensional submanifold.*

4.2.4 Method of the Hamiltonian Reduction

The *method of the Hamiltonian reduction* reduces a G-invariant Hamiltonian system on a symplectic manifold M with a Poisson G-action to a Hamiltonian system on a manifold of a smaller dimension. Its application can be hampered by some singularities. In the present section we shall describe the most regular situation, considered firstly in full generality in [117].

Suppose that M and G satisfy Definition 4.5, μ is the corresponding momentum map and a level set $M_c := \mu^{-1}(c)$ for some $c \in \mathfrak{g}^*$ is a smooth submanifold. In particular the last assumption is valid when c is a regular value of the momentum map μ, but in some interesting cases the momentum map has no regular values at all. Note that due to (4.16) the map μ has no regular values if stationary subalgebras $\mathfrak{g}_x$ are nontrivial for all $x \in M$.[3]

The group G_c acts on the manifold M_c. Suppose that this action is *proper* (it means that preimages of all compact sets w.r.t. the map $G \times M \to M \times M$, $(g, x) \to (gx, x)$ are compact) and free.[4] Then the orbit space $\widetilde{M}_c := G_c \backslash M_c$ is endowed with a structure of a smooth manifold such that the canonical projection $\pi_1 : M_c \to \widetilde{M}_c$ is a smooth map [58]. This orbit space is called the *reduced space* of the symplectic space M w.r.t. the given Poisson G-action.

The space $\widetilde{M}_c$ is endowed also with a symplectic structure in the following way. Suppose that $\tilde{x} \in \widetilde{M}_c$, $x \in M_c$, $\pi_1(x) = \tilde{x}$ and $\xi, \zeta \in T_xM$, $\tilde{\xi} = d\pi_1(\xi)$, $\tilde{\zeta} = d\pi_1(\zeta)$. Define 2-form $\widetilde{\omega}$ on $\widetilde{M}_c$ by the formula:

$$\widetilde{\omega}(\tilde{\xi}, \tilde{\zeta}) = \omega(\xi, \zeta) .$$

This form is well defined. Indeed, the space $\ker d_x\pi_1$ is tangent to the G_c-orbit passing through the point x and coincides with the space $\tilde{\mathfrak{g}}(x) \cap \ker d_x\mu$, which

[3] Informally speaking, the bigger is the symmetry group, the more singular is the Hamiltonian reduction.

[4] If the regularity assumptions of the last two paragraphs are not valid, then the reduction procedure is called singular. The review of different approaches in the singular situation can be found in [136]. Note that an action of a compact Lie group is always proper.

is skew-orthogonal to the space $T_x M_c = \ker d_x\mu = \tilde{\mathfrak{g}}(x)^\omega$. Therefore, the value $\widetilde{\omega}(\tilde{\xi}, \tilde{\zeta})$ does not depend on the choice of $\xi \in d\pi_1^{-1}(\tilde{\xi})$ and $\zeta \in d\pi_1^{-1}(\tilde{\zeta})$. Since G_c acts on M_c by symplectomorphisms, the form $\widetilde{\omega}$ also does not depend on the choice of $x \in \pi_1^{-1}(\tilde{x})$.

The form $\widetilde{\omega}$ is nondegenerate. Indeed, if $\widetilde{\omega}(\tilde{\xi}, \tilde{\zeta}) = 0$, $\forall \tilde{\zeta} \in T_{\tilde{x}}\widetilde{M_c}$, then the space $d\pi_1^{-1}(\tilde{\xi})$ is skew-orthogonal to $T_x M_c = \tilde{\mathfrak{g}}(x)^\omega$ and therefore $d\pi_1^{-1}(\tilde{\xi}) \subset \tilde{\mathfrak{g}}(x) \cap T_x M_c$. This means that $d\pi_1^{-1}(\tilde{\xi})$ is tangent to G_c-orbit and $\tilde{\xi} = 0$.

The form $\widetilde{\omega}$ is closed due to $0 = (d\omega)|_{M_c} = d\left(\omega|_{M_c}\right) = d\left(d\pi_1^*\widetilde{\omega}\right) = d\pi_1^* d\widetilde{\omega}$ and the surjectivity of $d\pi_1$. Thus, $\widetilde{M_c}$ is a symplectic space.

Let h be a G-invariant Hamiltonian function on M. The vector field X_h is tangent to M_c due to Proposition 4.4 and its restriction $X_h|_{M_c}$ is invariant w.r.t. G_c-action. Therefore, the projection $\widetilde{X}_h$ of the field X_h onto $\widetilde{M_c}$ is well defined.

Let $\tilde{h}$ be the projection of h onto $\widetilde{M_c}$. The Hamiltonian system on $\widetilde{M_c}$ with the Hamiltonian function $\tilde{h}$ is called the *reduced Hamiltonian system*.

Proposition 4.7. *The vector field $\widetilde{X}_h$ is Hamiltonian w.r.t. the symplectic form $\widetilde{\omega}$ with the Hamiltonian function $\tilde{h}$, in other words it holds $\widetilde{X}_h = X_{\tilde{h}}$. The map*

$$\eta_c : \ h \to \tilde{h} \tag{4.22}$$

is an homomorphism of the Poisson algebra $C^\infty(M)^G$ of G-invariant functions on M in the Poisson algebra $C^\infty(\widetilde{M_c})$.

Proof. The relation $dh = \omega(\cdot, X_h)$ implies the relation $d\tilde{h} = \widetilde{\omega}(\cdot, \widetilde{X}_h)$ due to definitions of $\tilde{h}, \widetilde{\omega}$ and $\widetilde{X}_h$, that proves the first part of the proposition. Let $h_1, h_2 \in C^\infty(M)^G$, then

$$[\tilde{h}_1, \tilde{h}_2]_P = -\widetilde{\omega}\left(\widetilde{X}_{h_1}, \widetilde{X}_{h_2}\right) = -\omega\left(X_{h_1}, X_{h_2}\right) = [h_1, h_2]_P,$$

i.e., the map (4.22) is a homomorphism of Poisson algebras. □

4.3 Hamiltonian Systems on Cotangent Bundles

Here we shall specify the theory of two preceding sections for the special type of symplectic manifolds, which are important for applications.

4.3.1 Canonical Symplectic Structure on Cotangent Bundles

Let N be a smooth manifold and $M = T^*N$ the corresponding cotangent bundle. A general point x of M can be written in the form $x = (q, p)$, $q \in N, p \in T_q^*N$. Let $\pi_4 : T^*N \to N$ be the canonical projection. Define the *canonical* 1-*form* α by the formula

$$\alpha(\xi) = p(d\pi_4(\xi)),\ \xi \in T_x\left(T^*N\right)\ .$$

If $q^1, \dots, q^m$ are local coordinates on N and $p_1, \dots, p_m$ are corresponding coordinates on the fiber $T_q^* N$, then

$$\alpha = \sum_{i=1}^{m} p_i dq^i \ .$$

Obviously, the 2-form

$$\omega := d\alpha = \sum_{i=1}^{m} dp_i \wedge dq^i$$

on M is nondegenerate, closed and thus determines the *canonical symplectic structure on the cotangent bundle* M. Coordinates p_i, $i = 1, \dots, m$ on linear fibers are called *momenta*, corresponding to coordinates q^i, $i = 1, \dots, m$ on N. In these coordinates one has

$$X_h = \sum_{i=1}^{m} \left(\frac{\partial h}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial}{\partial p_i} \right) ,$$

and the corresponding Poisson brackets are

$$[f,h]_P = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial q^i} \frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial h}{\partial q^i} \right) , \ f,h \in C^\infty(M) \ . \tag{4.23}$$

Further, let G be a Lie group with a left action on N by diffeomorphisms $q \to \psi_g(q)$, $g \in G, q \in N$. This action induces the *cotangent lifted* G-action on $M = T^* N$ by diffeomorphisms:

$$x = (q,p) \to \widehat{\psi}_g := \Big(\psi_g(q), d\psi^*_{g^{-1}}(p) \Big) , \tag{4.24}$$

where $d\psi^*_{g^{-1}} : T_q^* N \to T^*_{\psi_g(q)} N$ is the codifferential of the map $\psi_{g^{-1}}$. Clearly, this action conserves the form α. Therefore, due to (4.3), it holds

$$d\left(\alpha(\widetilde{Y}) \right) + (d\alpha)(\widetilde{Y}, \cdot) = \pounds_{\widetilde{Y}} \alpha = 0, \ Y \in \mathfrak{g}, \tag{4.25}$$

where $\widetilde{Y}$ is the same as in Definition 4.5 w.r.t. action (4.24). Particulary, action (4.24) is symplectic, since $\pounds_{\widetilde{Y}} \omega = \pounds_{\widetilde{Y}} (d\alpha) = d\left(\pounds_{\widetilde{Y}} \alpha \right) = 0$.

Proposition 4.8. *The action* (4.24) *is Poisson. The corresponding momentum map is defined as*

$$\mu(x)(Y) = \alpha(\widetilde{Y})\Big|_x , \ x \in M, Y \in \mathfrak{g} \ .$$

Proof. For $Y \in \mathfrak{g}$ define a function h_Y on M by the formula $h_Y = \alpha(\widetilde{Y})$. Due to (4.25) one has $dh_Y = \omega(\cdot, \widetilde{Y})$, that means that the vector field $\widetilde{Y}$ is Hamiltonian with the Hamiltonian function h_Y.

It remains to show that the correspondence $Y \to \alpha(\widetilde{Y})$ is a homomorphism of Lie algebras. Due to Proposition 4.1 the function

$$\left[\alpha(\widetilde{Y}), \alpha(\widetilde{Z})\right]_P, \; Y, Z \in \mathfrak{g}$$

is a Hamiltonian one for the vector field

$$-\left[\widetilde{Y}, \widetilde{Z}\right] = \widetilde{[Y, Z]}.$$

The sign here is due to (2.1). Since $\alpha\left(\widetilde{[Y, Z]}\right)$ is also a Hamiltonian function for the same vector field, the function

$$\left[\alpha(\widetilde{Y}), \alpha(\widetilde{Z})\right]_P - \alpha\left(\widetilde{[Y, Z]}\right) \tag{4.26}$$

is a constant. Due to (4.23) expression (4.26) is a homogeneous polynomial in momenta of the first degree. Thus, expression (4.26) vanishes, Q.E.D. □

4.3.2 Invariant Functions on Cotangent Bundles

Denote by $\mathcal{P}(T^*N) \subset C^\infty(M)$ the Poisson algebra, consisting of smooth real-valued functions, polynomial on fibers of the cotangent bundle T^*N.

Suppose now that N is homogeneous w.r.t. a left G-action and denote by $\mathcal{P}(T^*N)^G \subset \mathcal{P}(T^*N)$ the subalgebra of G-invariants w.r.t. the action (4.24). The theory of Sect. 2.1 for the algebra $\mathrm{Diff}_G(N)$ is transformed *mutatis mutandis* for the Poisson algebra $\mathcal{P}(T^*N)^G$.

Firstly consider the case $N = G$. Since the group G freely acts by left shifts on itself, the algebra $\mathcal{P}(T^*G)^G$ is naturally identified with the algebra $\mathcal{P}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$, i.e., with the symmetric algebra $S(\mathfrak{g})$. Also, the homomorphic extension $\imath_c$ of the linear map

$$Y^* \to Y^\# \in \mathcal{P}(T^*G)^G, \; Y^\#(q,p) = -p(d\pi_4(\widetilde{Y}^l)), \; q \in G, p \in T^*_qG \tag{4.27}$$

is the isomorphism of commutative algebras $S(\mathfrak{g})$ and $\mathcal{P}(T^*G)^G$, where $Y^* \in S(\mathfrak{g})$ is the element, corresponding to an element $Y \in \mathfrak{g}$, and $\widetilde{Y}^l$ is the corresponding left invariant vector field on T^*G. In other words, the vector field $\widetilde{Y}^l$ is generated by the element Y w.r.t. the cotangent lifted action of the group G on itself from the right:

$$(q, p) \to (qq_1, dR^*_{q_1^{-1}}p), \; q, q_1 \in G, p \in T^*_qG. \tag{4.28}$$

The isomorphism $\imath_c$ is the composition of the automorphism of $S(\mathfrak{g})$, induced by multiplication elements from $S_1(\mathfrak{g}) = \mathfrak{g}$ by -1, and the above identification $S(\mathfrak{g})$ with $\mathcal{P}(T^*G)^G$.

Proposition 4.9 (cf. Theorem 13.1.1 in [116]). *The map $\imath_c$ is an isomorphism of Poisson algebras $\mathcal{P}(T^*G)^G$ and $S_p(\mathfrak{g})$, where the Poisson structure for the latter was defined at the beginning of Sect.* 4.2.

Proof. Arguing as in the proof of Proposition 4.8 for the case of the G-action on T^*G from the right, one proves that the function $-Y^\#$ is a Hamiltonian one for a vector field $\widetilde{Y}^l$. Due to Proposition 4.1 the function

$$[Y^{\#}, Z^{\#}]_P,\ Y, Z \in \mathfrak{g}$$

is a Hamiltonian one for the vector field

$$-[\widetilde{Y}^l, \widetilde{Z}^l] = -\widetilde{[Y, Z]}^l\ .$$

Another Hamiltonian function for this vector field is $[Y, Z]^{\#}$ and due to the arguments in the proof of Proposition 4.8 it holds

$$[Y, Z]^{\#} = [Y^{\#}, Z^{\#}]_P\ .$$

This means that $\imath_c$ is the isomorphism of Poisson algebras $S_p(\mathfrak{g})$ and $\mathcal{P}(T^*G)^G$. □

Using Theorem 2.2 one gets the following isomorphism of Poisson algebras

$$\mathcal{P}(T^*G)^G \cong S_p(\mathfrak{g}) \cong \operatorname{gr} U(\mathfrak{g}) \cong \operatorname{gr} \operatorname{LDiff}(G)\ . \tag{4.29}$$

Let now $N \cong G/K$ be a reductive homogeneous space, where K is a stationary subgroup for a point $x_0 \in M$. As in Sect. 2.1.3 let $\mathfrak{p}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$ and $\mathrm{Ad}_K\, \mathfrak{p} \subset \mathfrak{p}$.

Evidently, due to the Leibniz rule the set $S(\mathfrak{g})^K$ is a Poisson subalgebra in $S_p(\mathfrak{g})$. Denote it by $S_p(\mathfrak{g})^K$.

Definition 4.6. *Call a linear subspace $\mathcal{A}'$ of a Poisson algebra $\mathcal{A}$ a* Poisson ideal of $\mathcal{A}$ *iff it is an ideal w.r.t. (commutative) multiplication and $[\mathcal{A}', \mathcal{A}]_P \subset \mathcal{A}'$.*

Let $S(\mathfrak{g})\mathfrak{k}$ be an ideal in commutative algebra $S(\mathfrak{g})$, generated by elements from $\mathfrak{k}$. Obviously, (cf. (2.9))

$$S(\mathfrak{g}) = S(\mathfrak{g})\mathfrak{k} \oplus S(\mathfrak{p}). \tag{4.30}$$

Let $(S(\mathfrak{g})\mathfrak{k})^K$ be a subalgebra in $S(\mathfrak{g})\mathfrak{k}$, consisting of Ad_K-invariant elements. It is a Poisson ideal in the Poisson algebra $S_p(\mathfrak{g})^K$, since for $f \in \mathfrak{k}$ and $g \in S_p(\mathfrak{g})^K$ one has $[f, g]_P = 0$ and thus $\left[(S_p(\mathfrak{g})\mathfrak{k})^K, g\right]_P \subset (S_p(\mathfrak{g})\mathfrak{k})^K$. Since both direct summands in (4.30) are Ad_K-invariant, one has

$$S(\mathfrak{g})^K = (S(\mathfrak{g})\mathfrak{k})^K \oplus S(\mathfrak{p})^K\ . \tag{4.31}$$

Therefore, the Poisson factor algebra $S_p(\mathfrak{g})^K / (S_p(\mathfrak{g})\mathfrak{k})^K$ is well defined and it is isomorphic as a commutative algebra to $S(\mathfrak{p})^K$, cf. Theorem 2.3.

The cotangent space $T^*_{x_0}N$ is naturally identified with $\operatorname{ann}\mathfrak{k} \subset \mathfrak{g}^*$. Elements from $\operatorname{ann}\mathfrak{k}$ are in one-to-one correspondence with elements from $\mathfrak{p}^*$. This correspondence is given by the restriction of an element $\phi \in \operatorname{ann}\mathfrak{k}$ onto $\mathfrak{p}$.

Due to the transitivity of G-action on N a function $f \in \mathcal{P}(T^*N)^G$ is uniquely determined by its restriction onto $T^*_{x_0}N \cong \mathfrak{p}^*$, which is Ad^*_K-invariant. On the other hand, if a polynomial ϕ on $\mathfrak{p}^* \cong T^*_{x_0}N$ is Ad^*_K-invariant, then it is uniquely extended to some function from $\mathcal{P}(T^*N)^G$.

Thus, one established the isomorphism between elements from $\mathcal{P}(T^*N)^G$ and invariants of the Ad_K^*-action on $\mathfrak{p}^*$. Clearly, latter invariants are in one-to-one correspondence with element from the set $S(\mathfrak{p})^K$, consisting of Ad_K-invariant elements from $S(\mathfrak{p})$.

Now we shall describe the Poisson algebra structure on $\mathcal{P}(T^*N)^G$ through the Poisson structure on $S_p(\mathfrak{g})$. Consider the following left K-action on the Lie group G:

$$g \to \chi_q(g) := gq^{-1},\ g \in G, q \in K \tag{4.32}$$

and its cotangent lift onto T^*G

$$x = (g,p) \to \left(\chi_q(g), d\chi^*_{q^{-1}}(p)\right),\ g \in G, p \in T^*_g G\,. \tag{4.33}$$

Due to Proposition 4.8 this action is Poisson. The zero level set of the corresponding momentum map is

$$(T^*G)_0 = \left((g,p) \in T^*G |\, g \in G, p \in \operatorname{ann}\mathfrak{k}_g \subset T^*_g G\right),$$

where $\mathfrak{k}_g$ is the image of the imbedding $\mathfrak{k}$ into T_gG by the differential of action (4.32). Evidently, $(T^*G)_0$ is a smooth subbundle of the cotangent bundle T^*G. The stationary subgroup of the element $0 \in \mathfrak{k}^*$ is the whole K and it acts on $(T^*G)_0$ freely and properly. The reduced phase space is identified with $T^*(G/K)$. It is easily seen that the reduced symplectic structure on $T^*(G/K)$ coincides with the canonical symplectic structure of a cotangent bundle.

Let $\mathcal{P}_K(T^*G)$ be a subalgebra of the commutative algebra $\mathcal{P}(T^*G)$, consisting of K-invariants of action (4.33). Moreover, since action (4.33) conserves the symplectic structure, $\mathcal{P}_K(T^*G)$ is a Poisson subalgebra of $\mathcal{P}(T^*G)$. Proposition 4.7 implies that the projection map

$$\eta_c : \mathcal{P}_K(T^*G) \to \mathcal{P}\left(T^*(G/K)\right),$$

constructed in Sect. 4.2.4, is a homomorphism of Poisson algebras. Considering the restriction of functions from $\mathcal{P}_K(T^*G)^G$ onto T^*_eG one obtains that the restriction $\bar{\eta}_c$ of the homomorphism η_c onto the Poisson subalgebra of G-invariants is the epimorphism:

$$\bar{\eta}_c : \mathcal{P}_K(T^*G)^G \to \mathcal{P}(T^*(G/K))^G\,.$$

Here the superscript G denotes the corresponding subalgebras of G-invariants w.r.t. left G-shifts on T^*G and $T^*(G/K)$. Similarly, the restriction $\bar{\imath}_c$ of the map $\imath_c$ (see (4.27)) onto $S_p(\mathfrak{g})^K$ is the isomorphism of Poisson algebras $S_p(\mathfrak{g})^K$ and $\mathcal{P}_K(T^*G)^G$.

Thus, one gets the sequence

$$S_p(\mathfrak{g})^K \xrightarrow{\ \bar{\imath}_c\ } \mathcal{P}_K(T^*G)^G \xrightarrow{\ \bar{\eta}_c\ } \mathcal{P}(T^*(G/K))^G$$

of homomorphisms of Poisson algebras.

Theorem 4.5 (cf. Theorem 2.3). *The Poisson algebra $\mathcal{P}(T^*(G/K))^G$ is isomorphic to the factor algebra $S_p(\mathfrak{g})^K/\left(S_p(\mathfrak{g})\mathfrak{k}\right)^K$ and also to the graded algebra for the filtered algebra $\mathrm{Diff}_G(G/K)$. The map*

$$\flat := \bar{\eta}_c \circ \bar{\imath}_c|_{S(\mathfrak{p})^K} : S(\mathfrak{p})^K \to \mathcal{P}(T^*(G/K))^G$$

is an isomorphism of commutative algebras.

Proof. Since $\bar{\imath}_c$ is isomorphism and $\bar{\eta}_c$ is epimorphism, one obtains that

$$\mathcal{P}(T^*(G/K))^G \cong S_p(\mathfrak{g})^K / \bar{\imath}_c^{-1}(\ker \bar{\eta}_c) \ .$$

Due to the definition of the map $\bar{\eta}_c$ the set $\ker \bar{\eta}_c$ consists of functions vanishing while being restricted onto $\operatorname{ann} \mathfrak{k}_e = \operatorname{ann} \mathfrak{k}$. Therefore $\bar{\imath}_c^{-1}(\ker \bar{\eta}_c) = (S_p(\mathfrak{g})\mathfrak{k})^K$ that proves the first claim.

Since the Ad_K-action conserves the filtration of $U(\mathfrak{g})$, the isomorphism

$$\operatorname{gr}\left(\mathrm{Diff}_G(G/K) \cong \mathcal{P}(T^*(G/K))^G\right) \tag{4.34}$$

follows from Theorems 2.1 and 2.3. The last claim of the proposition is the direct consequence of the first one and expansion (4.31). □

Remark 4.3. *If generators and corresponding relations for the algebra* $\mathrm{Diff}_G(G/K)$ *were constructed from homogeneous generators and relations in* $S(\tilde{\mathfrak{p}})^K$ *as described in Sect.* 2.1.4*, then one can obtain generators and relations for the Poisson algebra* $\mathcal{P}(T^*(G/K))^G$ *in the following way.*

Replace generators D_k *of the algebra* $\mathrm{Diff}_G(G/K)$ *by generators* p_k *of* $\mathcal{P}(T^*(G/K))^G$ *and obtain the corresponding relations for* p_k *simply by rejecting the lower terms in relations for* D_k *in* $\mathrm{Diff}_G(G/K)$*. Namely, in a commutator relation of the form* $[D_i, D_j] = P(D_k)$ *one should reject on the right hand side terms with degrees lower than* $\deg D_i + \deg D_j - 1$ *and obtain the corresponding relation in* $\mathcal{P}(T^*(G/K))^G$ *in the form* $[p_i, p_j]_P = \widetilde{P}(p_k)$*, where* $\widetilde{P}(D_k)$ *is the sum of monomials from* $P(D_k)$ *with total degrees equal to* $\deg D_i + \deg D_j - 1$.

The space G/K is called *weakly commutative* if the Poisson algebra $\mathcal{P}(T^*(G/K))^G$ is commutative. From Theorem 4.5 we see that if G/K is commutative, then it is weakly commutative. The inverse proposition was proved in [150].

Remark 4.4. *From remark* 4.2 *it follows that for a weakly commutative space* G/K *any function from* $\mathcal{P}(T^*(G/K))^G$ *corresponds to a Hamiltonian system, integrable in noncommutative sense modulo Assumption 4.1. It was shown in* [120] *that if any function from* $\mathcal{P}(T^*(G/K))^G$ *corresponds to a Hamiltonian system on* $T^*(G/K)$*, integrable in commutative sense due to integrals generated by the momentum map, then the homogeneous space* G/K *is weakly commutative.*

4.3.3 Natural Mechanical Systems and Dequantization

The transition from a commutative algebra of observables to a noncommutative one is called *quantization.* Usually there is more or less arbitrariness in this procedure. The geometric approach to quantization was described in

[57, 180, 209]. In the situation under consideration the most natural quantum analogue of the commutative algebra $\mathcal{P}(T^*(G/K))^G \cong S(\mathfrak{p})^K$ is the algebra $\mathrm{Diff}_G(G/K)$. From the symmetry point of view, the inverse map to (2.6): $\varkappa^{-1} : S(\mathfrak{p})^K \to \mathrm{Diff}_G(G/K)$ seems to be the most natural quantization map, transforming Hamiltonian functions into quantum mechanical Hamiltonians.

However, this map is not satisfactory from the physical point of view, since it can give Hamiltonians, which are not (even formally) self-adjoint and (or) has an improper spectrum. In the following we will restrict ourselves with so called *natural mechanical systems* on Riemannian spaces and describe for them the procedure of *dequantization*, which will admit to derive the classical mechanical expressions from quantum ones without tedious calculations.

Let N be a smooth manifold with local coordinates x^i on a domain $U \subset N$ and p_i be coordinates on fibers of T^*U, corresponding to coordinates x^i. Let D be a differential operator on N. Define the function $\mathrm{symb}\, D \in \mathcal{P}(T^*U)$ as the result of substituting p_i instead of operators $\partial/\partial x^i$ in a sum of leading monomials from an expression of D through local coordinates. Due to the tensor character of a coordinate change in leading terms of a differential operator the function $\mathrm{symb}\, D$ does not depend on the choice of local coordinates and thus is well defined on the whole space T^*N. Call $\mathrm{symb}\, D$ the *symbol of a differential operator* D. Obviously, $\mathrm{symb}\, D$ is a homogeneous polynomial on each fiber of T^*N, may be vanishing. If it is nontrivial on some fiber, then its degree equals $\deg D$.

Again, due to the tensor character of a coordinate change in leading terms of a differential operator, one gets the same function $\mathrm{symb}\, D$, operating in the similar way with the expression of D through a moving frames.

Suppose now that the manifold N, not necessarily homogeneous, is endowed with a Riemannian metric g, given in local coordinates by the expression $g_{ij}dx^i dx^j$. The Laplace-Beltrami operator $\triangle_g$, corresponding to this metric, is defined in (2.16). A *natural one-particle quantum mechanical Hamiltonian* is the operator

$$H = -\frac{1}{2m} \triangle_g + V \ , \tag{4.35}$$

where a function V on the space N is a potential. The Hamiltonian for many particle quantum mechanical system is the differential operator

$$\widetilde{H} = -\frac{1}{2} \sum_i \frac{\triangle_i}{m_i} + \widetilde{V} \ , \tag{4.36}$$

on the Riemannian space $\widetilde{N} = N \times \ldots \times N$, where $\triangle_i$ is the Laplace-Beltrami operator on the ith factor of $\widetilde{N}$, m_i is a mass of ith particle and a function $\widetilde{V}$ is an interactive potential.

Remark 4.5. *The operator* (4.36) *can be considered as the operator* (4.35) *w.r.t. the metric*

$$\tilde{g} = \prod_i m_i g_i$$

on $\widetilde{N}$, where g_i is the metric on ith factor of $\widetilde{N}$ and $m = 1$.

The *natural classical mechanical Hamiltonian function*, corresponding to (4.35), is the function

$$h = \frac{1}{2m} \sum_{i,j} g^{ij} p_i p_j + V \circ \pi_4 \;, \tag{4.37}$$

on T^*N, where $g^{ij} g_{jk} = \delta^i_k$. Clearly, it holds

$$h = \frac{1}{2m} \operatorname{symb} \triangle_g + V \circ \pi_4 \;. \tag{4.38}$$

The transition from quantum mechanical many particle systems to classical ones is similar due to remark 4.5.

Let G be a connected Lie group acting by isometries on N and $\mathcal{O} \subset N$ be a G-orbit. Suppose that some neighborhood of $\mathcal{O}$ in N is isometric to a direct product $W \times \mathcal{O}$ for some submanifold $W \subset N$, transversal to $\mathcal{O}$ (see Sect. 2.4), such that the G-action on N corresponds to its action only on the second factor of this product. Then the operator $\triangle_g|_{W\times\mathcal{O}}$ can be expressed in the form

$$\triangle_g|_{W\times\mathcal{O}} = \sum_k d_{k,0} \Box_{k,2} + \sum_k D_{k,1} \Box_{k,1} + D_2 + \sum_k \tilde{d}_k \widetilde{\Box}_k \;.$$

Here $d_{k,0}, \tilde{d}_k$ are smooth functions on W; $D_{k,1}$ are differential operators of the first order on W; D_2 is a differential operator of the second order on W; $\Box_{k,1}, \widetilde{\Box}_k$ are first order G-invariant differential operators on $\mathcal{O}$; $\Box_{k,2}$ are second order G-invariant differential operators on $\mathcal{O}$.

Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the same expansion as in Sects. 2.1.3 and 4.3.2 w.r.t. the homogeneous manifold $\mathcal{O}$ and the point $x_0 = W \cap \mathcal{O}$. Define the function from $\mathcal{P}(T^*(W \times \mathcal{O}))$ by the formula

$$\widetilde{\operatorname{symb}} \triangle_g = \sum_k d_{k,0} \cdot \flat \circ \varkappa_2(\Box_{k,2}) - \sum_k \operatorname{symb}(D_{k,1}) \cdot \flat \circ \varkappa_1(\Box_{k,1}) + \operatorname{symb} D_2,$$

where the map $\flat$ is defined in Theorem 4.5 and $\varkappa_k$, $k = 1, 2$ is the composition of the map $\varkappa$ (see (2.6)) with the projection of the algebra

$$S(\mathfrak{p}) = \bigoplus_{k=1}^{\infty} S_k(\mathfrak{p})$$

onto its direct summand $S_k(\mathfrak{p})$ consisting of homogeneous polynomials of k order (see Sect. 2.1.2).

Proposition 4.10. *It holds* $\operatorname{symb} \triangle_g = \widetilde{\operatorname{symb}} \triangle_g$ *on* $T^*(W \times \mathcal{O})$.

Proof. Since $\operatorname{symb}(D_1 \circ D_2) = \operatorname{symb} D_1 \operatorname{symb} D_2$ for any differential operators D_1 and D_2, it is enough to prove that $\operatorname{symb}(\Box_{k,2}) = \flat \circ \varkappa_2 (\Box_{k,2})$ and $\operatorname{symb}(\Box_{k,1}) = -\flat \circ \varkappa_1 (\Box_{k,1})$ on $T^*\mathcal{O}$.

Let e_i be a base in $\mathfrak{p}$ and $\tilde{e}_i$ be the corresponding vector fields on $\mathcal{O}$, forming a moving frame in some neighborhood U of the point $x_0 \in \mathcal{O}$. Let

$$\Box_{k,2} = \sum_{i,j} \hat{g}_k^{ij} \tilde{e}_i \circ \tilde{e}_j + \sum_i \tilde{g}^i \tilde{e}_i,\ \Box_{k,1} = \sum_i \hat{g}^i \tilde{e}_i$$

be expressions for $\Box_{k,2}$ and $\Box_{k,1}$ in U. Then by definition of $\varkappa$ one has

$$\varkappa_2(\Box_{k,2}) = \sum_{i,j} \left.\hat{g}_k^{ij}\right|_{x_0} e_i^* e_j^*,\ \varkappa_1(\Box_{k,1}) = \sum_i \left.\hat{g}^i\right|_{x_0} \tilde{e}_i^* \ .$$

By definition of the map $\flat$ in Theorem 4.5 one gets

$$\flat \circ \varkappa_2(\Box_{k,2})|_{x_0} = \sum_{i,j} \hat{g}^{ij} \tilde{p}_i \tilde{p}_j\big|_{x_0},\ \ \flat \circ \varkappa_1(\Box_{k,1})|_{x_0} = -\sum_i \hat{g}^i \tilde{p}_i\big|_{x_0},$$

where $\tilde{p}_i$ are coordinates w.r.t. the base $\tilde{e}^i$, dual to $\tilde{e}_i$. Due to the definition of the symbol map this completes the proof. □

4.3.4 Reduction of Cotangent Bundles of Homogeneous Manifolds

Suppose as before that $N = G/K$ is a G-homogeneous space with the reductive expansion $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, $\pi_1 : G \to N$ is the canonical projection and $M = T^*N$ is the cotangent bundle for N. The space M is endowed with the Poisson G-action $\widehat{\psi}_G$ (see (4.24)) and the momentum map $\mu :\ M \to \mathfrak{g}^*$, defined in Proposition 4.8. Let $\pi_4 : T^*N \to N$ be another canonical projection. Identify the tangent space $T_{x_0}N$ with $\mathfrak{p}$ and the space $T^*_{x_0}N$ with $\mathfrak{p}^* \cong \operatorname{ann}\mathfrak{k} \subset \mathfrak{g}^*$. Under such identification the differential of left K-shifts on $T_{x_0}N$ becomes the Ad_K-action on $\mathfrak{p}$ and the differential $d\widehat{\psi}_K$ on $T^*_{x_0}N$ becomes the Ad^*_K-action on $\mathfrak{p}^*$. The reduced phase space for T^*N under some regularity assumption admits the description in terms of coadjoint orbits of the symmetry group G [163, 165].

Consider a level set $M_\beta := \mu^{-1}(\beta) \subset M$ for $\beta \in \operatorname{im}\mu$. Without loss of generality one can suppose that $T^*_{x_0}N \cap M_\beta \neq \varnothing$, since N is G-homogeneous. This means that $T^*_{x_0}N \cap M_\beta = (x_0, \beta)$, $\beta \in \operatorname{ann}\mathfrak{k} \cong \mathfrak{p}^*$. The stationary subgroup of the group G, corresponding to the point (x_0, β), is the stationary subgroup $K_0 \subset K$ of the element $\beta \in \mathfrak{p}^* \cong \operatorname{ann}\mathfrak{k}$ w.r.t. the $\operatorname{Ad}^*_K|_{\mathfrak{p}^*}$-action. Evidently, elements $y \in M_\beta$ are in one-to-one correspondence with elements of the form $\pi_4(y)$. Denote $M'_\beta = \pi_4(M_\beta)$.

Identify the algebra $\mathfrak{g}$ with the tangent space T_eG and the space $\mathfrak{g}^*$ with the space T^*_eG. As above, denote by X^r the right invariant vector field on G, corresponding to an element $X \in \mathfrak{g}$. Consider a general point of M_β in the form $x = (\pi_1 g, p') \in M_\beta$, $g \in G, p' \in T^*_{\pi_1 g}N$ and the corresponding point $(g, p) \in T^*G$, where $p = d\pi_1^*(p')$. According to the definition of the momentum map it holds $p(X^r|_g) = \beta(X)$, i.e., $p = dR^*_{g^{-1}}\beta$. Also, $X^r|_g \in \mathcal{N} := \ker d\pi_1|_{T_gG}$ iff $X \in \operatorname{Ad}_g \mathfrak{k}$. Since $p|_{\mathcal{N}} = 0$, one has $\beta \in \operatorname{ann}\operatorname{Ad}_g\mathfrak{k}$ or equivalently $\operatorname{Ad}^*_{g^{-1}}\beta \in \operatorname{ann}\mathfrak{k}$.

Conversely, if $\operatorname{Ad}^*_{g^{-1}}\beta \in \operatorname{ann}\mathfrak{k}$, then $dR^*_{g^{-1}}\beta\Big|_{\mathcal{N}} = 0$ and there exists a unique element $p' \in T^*_{\pi_1 g}N$ such that $p := d\pi_1^*(p') = dR^*_{g^{-1}}\beta$ and $(\pi_1 g, p') \in M_\beta$. Denote

$$\mathcal{G}_\beta = \left(g \in G \mid \operatorname{Ad}^*_{g^{-1}}\beta \in \operatorname{ann}\mathfrak{k}\right) \ .$$

Assumption 4.2. *Suppose that $\mathcal{G}_\beta$ is a nonempty smooth submanifold of G.*

Obviously, this assumption is the corollary of

Assumption 4.3. *The orbit $\mathcal{O}_\beta$ of the coadjoint action Ad^*_G of the group G in $\mathfrak{g}^*$, containing a point $\beta \in \operatorname{ann}\mathfrak{k}$, is transversal to the subspace $\operatorname{ann}\mathfrak{k} \subset \mathfrak{g}^*$.*

Clearly, the group K acts on $\mathcal{G}_\beta$ by right shifts freely and properly. Orbits of this action are identified with elements of M'_β. Therefore, under assumption 4.2, the set M'_β is a smooth manifold, diffeomorphic to M_β. Denote by

$$\varpi : \ \mathcal{G}_\beta \to M'_\beta \cong \mathcal{G}_\beta / R_K$$

the corresponding right principle bundle. In fact $\varpi = \pi_1|_{\mathcal{G}_\beta}$.

Let G_β be the stationary subgroup of the group G w.r.t. the Ad^*_G-action and the point $\beta \in \mathfrak{g}^*$. It acts on $\mathcal{G}_\beta$ by left shifts freely and properly. Orbits of this action are in one to one correspondence with level sets of the map

$$\tau : \ \mathcal{G}_\beta \to \operatorname{ann}\mathfrak{k}, \ \tau(g) = \mathrm{Ad}^*_{g^{-1}} \beta \, . \tag{4.39}$$

Let

$$\mathcal{O}'_\beta := \mathcal{O}_\beta \cap \operatorname{ann}\mathfrak{k} \, .$$

Thus, one gets the left principle bundle

$$\tau : \ \mathcal{G}_\beta \to \mathcal{O}'_\beta \cong \mathcal{G}_\beta / L_{G_\beta},$$

where the set $\mathcal{O}'_\beta$ is a smooth manifold under Assumption 4.2.

The right K-shifts on $\mathcal{G}_\beta$ and the left G_β-shifts on $\mathcal{G}_\beta$ commute and it holds

$$\tau(gq) = \mathrm{Ad}^*_{q^{-1}} \circ \tau(g), \ g \in \mathcal{G}_\beta, q \in K \, . \tag{4.40}$$

Thus, the group K acts on $\mathcal{O}'_\beta$ by the map $\mathrm{Ad}^*_{q^{-1}}$, $q \in K$.

Similarly, the group G_β acts on M'_β by the map

$$\varpi(g) \to q\varpi(g) = \varpi(qg), \ g \in \mathcal{G}_\beta, q \in G_\beta \, .$$

Denote by $\widetilde{\mathcal{O}}_\beta := \mathcal{O}'_\beta / \mathrm{Ad}^*_K = M'_\beta / G_\beta$ the corresponding factor space and by

$$\tau' : \ M'_\beta \to \widetilde{\mathcal{O}}_\beta, \ \varpi' : \ \mathcal{O}'_\beta \to \widetilde{\mathcal{O}}_\beta$$

the corresponding canonical projections. Thus, one obtains the commutative diagram:

$$\begin{array}{ccc} \mathcal{G}_\beta & \xrightarrow{\ \tau\ } & \mathcal{O}'_\beta \\ \varpi \big\downarrow & & \big\downarrow \varpi' \\ M'_\beta & \xrightarrow{\ \tau'\ } & \widetilde{\mathcal{O}}_\beta \end{array} \tag{4.41}$$

Assumption 4.4. *Suppose that Ad^*_K-action on $\mathcal{O}'_\beta$ is free and proper. Due to (4.40) the first condition is equivalent to the condition that an intersection of any R_K-orbit on $\mathcal{G}_\beta$ and any L_{G_β}-orbit on $\mathcal{G}_\beta$ consists of no more than one point. It is also equivalent to the condition that L_{G_β}-action on M'_β is free.*

Under assumptions 4.2 and 4.4 the factor space $\widetilde{\mathcal{O}}_\beta = \mathcal{O}'_\beta / \operatorname{Ad}^*_K$ is a smooth manifold and it is diffeomorphic to the reduced space $\widetilde{M}_\beta = M'_\beta / G_\beta$ from Sect. 4.2.4, endowed with the symplectic structure $\widetilde{\omega}$. Also, this symplectic structure can be obtained from the canonical symplectic structure on Ad^*_G-orbits.

Indeed, let ω^* be the restriction of the Kirillov form onto $\mathcal{O}'_\beta$ and β' be an arbitrary point of $\mathcal{O}'_\beta$. Let X^* and Y^* be vector fields on $\mathcal{O}_\beta$, corresponding to elements $X, Y \in \mathfrak{g}$ as in (4.10), such that $X^*|_{\beta'}, Y^*|_{\beta'} \in T_{\beta'}\mathcal{O}'_\beta$. Due to the definition of the Kirillov form in (4.11) it means that $\omega^*(X^*, Y^*)|_{\beta'} = -\beta'([X, Y])$. Since the vector $X^*|_{\beta'}$ is tangent to $\mathcal{O}'_\beta$, it holds

$$\left.\frac{d}{dt}\right|_{t=0} \left(\operatorname{Ad}^*_{\exp(tX)} \beta'\right) \in \operatorname{ann} \mathfrak{k},$$

and thus

$$\beta'([X, Y_0]) = -\left.\frac{d}{dt}\right|_{t=0} \left(\operatorname{Ad}^*_{\exp(tX)} \beta'\right)(Y_0) = 0$$

for any $Y_0 \in \mathfrak{k}$. This means that the following 2-form $\widehat{\omega}$ is well-defined on $\widetilde{O}_\beta$ by the formula:

$$\widehat{\omega}(\widetilde{X}, \widetilde{Y}) = \omega^*(d\varpi'^{-1}\widetilde{X}, d\varpi'^{-1}\widetilde{Y})$$

for $\widetilde{X}, \widetilde{Y} \in T_{\varpi'\beta'}\widetilde{\mathcal{O}}_\beta$.

Theorem 4.6. *Under assumptions* 4.2 *and* 4.4 *the reduced space $\widetilde{M}_\beta$, corresponding to a value β of the momentum map, is diffeomorphic to the space $\widetilde{\mathcal{O}}_\beta$. Moreover, the identification of $\widetilde{M}_\beta$ and $\widetilde{\mathcal{O}}_\beta$ by the map τ' gives $\widehat{\omega} = -\widetilde{\omega}$.*

Proof. The first claim is already proved. Let $\mathcal{G}'_\beta$ be the image of the inclusion of $\mathcal{G}_\beta$ into T^*G through the map

$$\jmath : \; g \to \left(g, dR^*_{g^{-1}}\beta\right) \; .$$

Let $\omega' = d\jmath^*(\omega|_{\mathcal{G}'_\beta})$ be the 2-form on $\mathcal{G}_\beta$, where $\omega = d\alpha$ for the canonical 1-form α on T^*G. Also, let α_N be the canonical 1-form on T^*N, the 2-form $\omega_N = d\alpha_N$ be the canonical symplectic structure on T^*N and ω'_N be the 2-form on M'_β, induced by ω_N through the isomorphism $\kappa := \pi_4|_{M_\beta} : \; M_\beta \to M'_\beta$.

Due to the construction of the 2-form $\widetilde{\omega}$ in Sect. 4.2.4 for the proof of the second claim it is enough to show that

$$d\tau^*(\omega^*) = -\omega', \; d\varpi^*(\omega'_N) = \omega' \; .$$

Let g be an arbitrary fixed point of $\mathcal{G}_\beta$. An arbitrary vector from $T_g\mathcal{G}_\beta$ can be represented in the form $X^l\big|_g$ for some $X \in \mathfrak{g}$, where X^l is the left invariant vector field on G. The map $d\tau$ transforms the vector field $X^l\big|_{\mathcal{G}_\beta}$ into the vector field $-X_c$ on $\mathcal{O}'_\beta$, see (4.10). Let $Y^l\big|_g$ be another vector from $T_g\mathcal{G}_\beta$ for some $Y \in \mathfrak{g}$. Let $\widetilde{X}^l$ and $\widetilde{Y}^l$ be vector fields on T^*G, induced by elements X and Y w.r.t the right G-action (4.28) on T^*G.

Clearly, $\left.\widetilde{X}^l\right|_{\jmath(g)} = d\jmath(X^l|_g) \in T_{\jmath(g)}\mathcal{G}'_\beta$, $\left.\widetilde{Y}^l\right|_{\jmath(g)} = d\jmath(Y^l|_g) \in T_{\jmath(g)}\mathcal{G}'_\beta$ and also due to (4.11) and (4.39) one gets

$$\omega^*\left(d\tau(X^l|_g), d\tau(Y^l|_g)\right) = \omega^*\left(X_c, Y_c\right)|_{\tau(g)} = -\left(\mathrm{Ad}^*_{g^{-1}}\beta\right)([X,Y]) \ . \quad (4.42)$$

On the other hand, by definition of the 2-form ω it holds

$$\omega\left(\widetilde{X}^l, \widetilde{Y}^l\right) = d\alpha\left(\widetilde{X}^l, \widetilde{Y}^l\right) = \widetilde{X}^l\left(\alpha\left(\widetilde{Y}^l\right)\right) - \widetilde{Y}^l\left(\alpha\left(\widetilde{X}^l\right)\right) - \alpha\left(\left[\widetilde{X}^l, \widetilde{Y}^l\right]\right) \ .$$

Since vector fields $\widetilde{X}^l$ and $\widetilde{Y}^l$ generate flows conserving the form α, one has

$$\widetilde{X}^l\left(\alpha\left(\widetilde{Y}^l\right)\right) = \alpha\left(\pounds_{\widetilde{X}^l}\widetilde{Y}^l\right) = \alpha\left(\left[\widetilde{X}^l, \widetilde{Y}^l\right]\right) = \alpha\left(\widetilde{[X,Y]}^l\right)$$

due to (2.3) and similarly

$$\widetilde{Y}^l\left(\alpha\left(\widetilde{X}^l\right)\right) = \alpha\left(\widetilde{[Y,X]}^l\right) \ .$$

Thus, due to definitions of ω' and α, it holds

$$\begin{aligned}\omega'\left(X^l|_g, Y^l|_g\right) &= \omega\left(\widetilde{X}^l, \widetilde{Y}^l\right)\Big|_{(g, dR^*_{g^{-1}}\beta)} = \alpha\left(\widetilde{[X,Y]}^l\right)\Big|_{(g, dR^*_{g^{-1}}\beta)} \\ &= \left(dR^*_{g^{-1}}\beta\right)\left(dL_g\left([X,Y]\right)\right) = \left(dL^*_g \circ dR^*_{g^{-1}}\beta\right)([X,Y]) = \left(\mathrm{Ad}^*_{g^{-1}}\beta\right)([X,Y]) \ .\end{aligned} \quad (4.43)$$

Therefore, from (4.42) one gets

$$\omega'\left(X^l|_g, Y^l|_g\right) = -\omega^*\left(d\tau(X^l|_g), d\tau(Y^l|_g)\right)$$

or equivalently $d\tau^*(\omega^*) = -\omega'$.

At the same time any vector from $T_g\mathcal{G}_\beta$ has the form $X^r|_g$ for some $X \in \mathfrak{g}$, where X^r is the right invariant vector field on G. Let $Y^r|_g$ be another such vector. Evidently, $X^r|_g = \left.\left(\mathrm{Ad}_{g^{-1}}X\right)^l\right|_g$, $Y^r|_g = \left.\left(\mathrm{Ad}_{g^{-1}}Y\right)^l\right|_g$ and (4.43) implies

$$\begin{aligned}\omega'\left(X^r|_g, Y^r|_g\right) &= \left(\mathrm{Ad}^*_{g^{-1}}\beta\right)\left(\left[\mathrm{Ad}_{g^{-1}}X, \mathrm{Ad}_{g^{-1}}Y\right]\right) \\ &= \beta\left(\mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}}[X,Y]\right) = \beta\left([X,Y]\right) \ .\end{aligned} \quad (4.44)$$

Let $X_N = d\pi_1(X^r), Y_N = d\pi_1(Y^r) \in \mathcal{X}(N)$ and

$$\widetilde{X}_N = (d\kappa)^{-1}\left(X_N|_{M'_\beta}\right) \in \mathcal{X}(M)|_{M_\beta} \ , \ \widetilde{Y}_N = (d\kappa)^{-1}\left(Y_N|_{M'_\beta}\right) \in \mathcal{X}(M)|_{M_\beta} \ ,$$

where $\mathcal{X}(M)|_{M_\beta}$ denotes the restriction of the module $\mathcal{X}(M)$ onto $M_\beta \subset M$. Note that vector fields X_N, Y_N correspond to left G-shifts on N (cf. Sect. 2.1.1) and one can define a vector field Z_N for an arbitrary $Z \in \mathfrak{g}$. Likewise (2.2), it holds

$$[X_N, Y_N] = -[X, Y]_N \ . \tag{4.45}$$

Let $\mathbf{x} := (\varpi(g), p') := \kappa^{-1} \circ \varpi(g)$, $p' \in T^*_{\varpi(g)}N$. From $X^r|_g, Y^r|_g \in T_g\mathcal{G}_\beta$ one gets $X_N|_g, Y_N|_g \in T_{\varpi(g)}M'_\beta$ and therefore

$$\widetilde{X}_N\Big|_{\mathbf{x}} \in T_{\mathbf{x}}(M_\beta), \ \widetilde{Y}_N\Big|_{\mathbf{x}} \in T_{\mathbf{x}}(M_\beta) \ . \tag{4.46}$$

At other points $\mathbf{y} \neq \mathbf{x}$ of M_β it is possible that $\widetilde{X}_N\Big|_y, \widetilde{Y}_N\Big|_y \notin T_{\mathbf{y}}(M_\beta)$. In any case due to the definition of ω'_N

$$\begin{aligned}\omega'_N(X_N, Y_N)|_{\varpi(g)} = \omega_N\left(\widetilde{X}_N, \widetilde{Y}_N\right)\Big|_{\mathbf{x}} &= \widetilde{X}_N\left(\alpha_N\left(\widetilde{Y}_N\right)\right)\Big|_{\mathbf{x}} \\ &- \widetilde{Y}_N\left(\alpha_N\left(\widetilde{X}_N\right)\right)\Big|_{\mathbf{x}} - \alpha_N\left(\left[\widetilde{X}_N, \widetilde{Y}_N\right]\right)\Big|_{\mathbf{x}} \ .\end{aligned}$$

Since functions

$$\alpha_N\left(\widetilde{Y}_N\right)\Big|_{M_\beta} = p'(Y_N)|_{M'_\beta} = \beta(Y), \ \ \alpha_N\left(\widetilde{X}_N\right)\Big|_{M_\beta} = \beta(X)$$

are constant on M_β, taking into account (4.46), one gets

$$\widetilde{X}_N\left(\alpha_N\left(\widetilde{Y}_N\right)\right)\Big|_{\mathbf{x}} = \widetilde{Y}_N\left(\alpha_N\left(\widetilde{X}_N\right)\right)\Big|_{\mathbf{x}} = 0 \ .$$

Therefore, it holds

$$\begin{aligned}\omega'_N(X_N, Y_N)|_{\varpi(g)} &= -p'\left(d\kappa\left(\left[\widetilde{X}_N, \widetilde{Y}_N\right]\right)\right)\Big|_{\varpi(g)} \\ &= -p'\left(\left[d\kappa\left(\widetilde{X}_N\right), d\kappa\left(\widetilde{Y}_N\right)\right]\Big|_{\varpi(g)}\right) \\ &= -p'\left([X_N, Y_N]|_{\varpi(g)}\right) = p'\left([X, Y]_N|_{\varpi(g)}\right) = \beta([X, Y])\end{aligned}$$

due to (4.45).

Thus, comparing the last formula with (4.44), one concludes that $\omega' = d\varpi^*(\omega'_N)$. This completes the proof. □

The form $\widetilde{\omega}$ is symplectic, so we get:

Corollary 4.1. *The form $\widehat{\omega}$ is symplectic on $\widetilde{\mathcal{O}}_\beta$, i.e., it is nondegenerate and closed.*

Remark 4.6. *This theorem generalizes the well-known fact that the reduced space for the symplectic space T^*G is symplectomorphic to an orbit of Ad^*_G-action [8, 117]. In this case $K = \{e\}$, $\mathcal{G}_\beta = G \cong M'_\beta$, $\widetilde{\mathcal{O}}_\beta = \mathcal{O}'_\beta = \mathcal{O}_\beta$ and assumptions 4.2, 4.4 are trivially fulfilled.*

In Sect. 7.5 we shall use the symplectomorphism from Theorem 4.6 for the reduction of the space T^*N, where N is a smooth manifold with a Lie group G acting on it, but not in a transitive way. Below there are some historical comments.

The Hamiltonian reduction of cotangent bundle T^*N, under assumption that G acts freely on N, was studied in [97, 114, 115] (see also references therein). The latter assumption is too restrictive at least for the two-body problem on a two-point homogeneous space Q with $\dim_{\mathbb{R}} Q \geqslant 3$. The approach of [97] to the cotangent reduction was generalized for a non-free G-action in preprint [126]. This paper gives the description of the reduced space, corresponding to $\beta \in \operatorname{Im}\mu \subset \mathfrak{g}^*$ as a subbundle P_β of the cotangent bundle $T^*(N_\beta/G_\beta)$, where N_β is the image of the canonical projection

$$T^*N \supset \mu^{-1}(\beta) \to N,$$

and G_β, as above, is the stationary subgroup of the group G, w.r.t. Ad^*_G-action. The symplectic structure on P_β is induced by a two-form on $T^*(N_\beta/G_\beta)$ that differs from the canonical symplectic structure on $T^*(N_\beta/G_\beta)$ by the additional "magnetic" term.

Our approach in Theorem 4.6 and its generalization for the non transitive G-action on N seem to be more straightforward, since they give the description of the reduced phase space only in terms of coadjoint orbits, the canonical symplectic structure on them and the factor space N/G.

5

Two-Body Hamiltonian on Two-Point Homogeneous Spaces

The main characteristic for the system of two classical particles on a Riemannian space M is the distance between them. If the space M is homogeneous and isotropic, this distance is the only geometric invariant for a position of particles in M. This motivates the separation of degrees of freedom into two types. The first type contains only one radial degree of freedom. The second one contains other degrees of freedom, which correspond to the isometry group. Such separation should appear itself also in studying the two-body quantum Hamiltonian (or the classical Hamiltonian function) and in an expansion of the corresponding Hilbert space.

In this chapter we derive a rigorous base for these separation in quantum case. Namely we find an expression of the two-body Hamiltonian through a radial differential operator and generators of the algebra $\mathrm{Diff}_I(Q_{\mathbf{S}})$ from Chap. 3.

The corresponding expression for the two-body classical Hamiltonian function can be derived in a similar way. But it is simpler to get such expression from its quantum analogue (provided it is already calculated), using Proposition 4.10 that will be done in Chap. 7.

Sections 5.1–5.3 follow the consideration in [169] with correction of some misprints.

5.1 Homogeneous Submanifolds in the Configuration Space of the Two-Body Problem

Let $\widetilde{M} = Q$ be an arbitrary two-point homogeneous Riemannian space with $\dim_{\mathbb{R}} Q = n$ and a space M be the direct product $Q \times Q$. Denote also by $\tilde{\pi}_i$, $i = 1, 2$ the projection onto the ith factor of the product $M = Q \times Q$, by G the identity component of the isometry group for Q and by $\rho(x_1, x_2)$ the distance between points $x_1, x_2 \in Q$. The function $\rho_2(x) := \rho(\tilde{\pi}_1(x)), \tilde{\pi}_2(x)), x \in M$ determines the distance between particles.

At last let $H = H_0 + V$ be the two-body Hamiltonian (2.40). The free Hamiltonian H_0 on the space M is the Laplace-Beltrami operator for the metric

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$$g_2 := m_1 \tilde{\pi}_1^* g + m_2 \tilde{\pi}_2^* g \tag{5.1}$$

on this space, multiplied by $-1/2$, where $\tilde{\pi}_i^* g$ is the pullback of the metric g with respect to the projection on the ith factor. In order to find an explicitly invariant expression for the operator H_0, consider the foliation of the space M by submanifolds F_p that are level sets of the function ρ_2. A fiber $F_p \subset M$ is G-homogeneous Riemannian manifolds with respect to the restriction of the metric g_2 on it; therefore, one can use the construction from Sect. 2.1 for the description of differential operators on this fiber. "To glue" these constructions for different p we shall proceeds as follows.

Suppose that Q is a compact two-point homogeneous Riemannian spaces. Let $\widetilde{\gamma}(s) : \mathbf{S}^1 \equiv \mathbb{R} \mod (2\,\mathrm{diam}\, Q) \to Q$ be some (evidently closed) geodesic on the space Q, where s is a natural parameter. Let also $s_1, s_2 : [0, \mathrm{diam}\, Q) \to \mathbf{S}^1 \equiv \mathbb{R} \mod (2\,\mathrm{diam}\, Q)$ be smooth functions such that s_1 is decreasing, s_2 is increasing, $s_1(0) = s_2(0) = 0$ and $\rho(\widetilde{\gamma}(s_1(p)), \widetilde{\gamma}(s_2(p))) \equiv p$, $p \in [0, \mathrm{diam}\, Q)$, $s_1'(p)^2 + s_2'(p)^2 \neq 0$. Define a curve $\gamma : I := (0, \mathrm{diam}\, Q) \to M$ by the formula $\gamma(p) = (\widetilde{\gamma}(s_1(p)), \widetilde{\gamma}(s_2(p))) \in M$. A stationary subgroup of the group G, corresponding to any point $\gamma(p)$, $0 < p < \mathrm{diam}\, Q$, is the stationary subgroup $K_0 \subset G$, corresponding to the pair $\widetilde{\gamma}(s_1(p)), \widetilde{\gamma}(s_2(p))$ of points on the geodesic $\widetilde{\gamma}$. Due to Proposition 1.1 the group K_0 does not depend on the choice of p if $0 < p < \mathrm{diam}\, Q$ and the factor space G/K_0 is isomorphic to the unit sphere bundle over Q as a G-homogeneous space. This construction is valid also for a noncompact two-point homogeneous Riemannian spaces if one assumes $\mathrm{diam}\, Q = \infty$.

This consideration can be summarized in the following lemma.

Lemma 5.1. *The map $p \to \gamma(p)$ from I to M is a regular curve γ in M, i.e., $|\gamma'(p)| \neq 0$ for all $p \in I$. This curve intersects each fiber F_p, $p > 0$ exactly at one point and the set $M' := \bigcup_{p \in I} F_p$ is a connected dense open submanifold in M. Stationary subgroups of the group G for the points $\gamma(p)$ coincide with each other for all $p \in I$.*

Thus, one can identify the manifold M' with the space $I \times (G/K_0)$ by the following formula:

$$I \times (G/K_0) \ni (p, bK_0) \longleftrightarrow b\gamma(p) \in M',\ b \in G\ .$$

Evidently, the complementary set $M \backslash M'$ for a compact Q is the disjoint union of the diagonal $\mathrm{diag}\,(Q \times Q)$ and the set $Q_{op} := F_{\mathrm{diam}\, Q}$. The latter set is the fiber bundle with the base Q and fibers equal to antipodal manifolds A_x, $x \in Q$. For a noncompact space Q the set $M \backslash M'$ coincides with $\mathrm{diag}\,(Q \times Q)$.

Let μ_2 be a measure on M, generated by the metric g_2. Using the identification above one can consider μ_2 also as a measure on the space $I \times (G/K_0)$. Since the set $M \backslash M'$ has a zero measure, we get the following isomorphism between spaces of measurable square integrable functions:

$$\mathcal{L}^2\left(M, \mu_2\right) \cong \mathcal{L}^2\left(I \times (G/K_0), \mu_2\right)\ . \tag{5.2}$$

In the following for simplicity it will be convenient to change the parametrization of the interval I using some function $p(r)$, $p'(r) \neq 0$, $r \in I' \subset \mathbb{R}_+$. In this case we will write $F_r := F_{p(r)}$. Since the group G acts only on the second factor of the expansion $M' = I \times (G/K_0)$, one can generalize the construction for the lift of differential operators from Sect. 2.1 and find for a G-invariant differential operator on the space $I \times (G/K_0)$ its lift onto the space $I \times G$.

Let $\mathfrak{p}$ be a subspace in $\mathfrak{g}$, complimentary to the subalgebra $\mathfrak{k}_0$ such that $[\mathfrak{p}, \mathfrak{k}_0] \subset \mathfrak{p}$. Let $e_1, \dots e_{2n-1}$ be a basis in $\mathfrak{p}$; $X_1, \dots, X_{2n-1}$ be the corresponding Killing vector fields on the space M' and X_i^l, X_i^r be the corresponding left- and right-invariant vector fields on the group G. Define a vector field tangent to the curve γ by the formula $X_0 = \frac{d}{dr}\gamma(p(r))$. Since

$$dL_q X_0 = \frac{d}{dr} L_q \gamma(p(r)) = \frac{d}{dr}\gamma(p(r)) = X_0,\ \forall q \in K_0\ ,$$

it is possible to spread the vector X_0 by left shifts to the whole space M' and obtain the smooth vector field on M' with the same notation X_0. The fields X_i, $i = 0, \dots, 2n-1$ form the moving frame in some neighborhood of the curve $\gamma(p)$, $p \in (0, \operatorname{diam} Q)$, if the matrix $\mathcal{B}$, consisting of the pairwise scalar products of the fields X_i, is nondegenerate on $\gamma(p)$, $p \in (0, \operatorname{diam} Q)$. The next condition will be verified later in Sect. 5.2.

Condition 5.1. *The matrix $\mathcal{B}$ is nonsingular on the curve $\gamma(p)$, $p \in (0, \operatorname{diam} Q)$.*

Express the operator $\triangle_{g_2}$ via the moving frame X_i, $i = 0, \dots, 2n-1$ by the formula (2.18), assuming $\xi_i = X_i$, $i = 0, \dots, 2n-1$, and transform the result to the form $a^{ij} X_i \circ X_j + b^i X_i$. Since the field X_0, in contrast to other fields X_i, is not a Killing one, after calculations similar to (2.20), we obtain the following additional terms:

$$-\left(\pounds_{X_0} g_2\right)(X_i, X_j)\hat{g}_2^{0i} + \frac{1}{2}\hat{g}_2^{ki}\left(\pounds_{X_0} g_2\right)(X_k, X_i)\delta_j^0\ ,$$

where $\hat{g}_{2,ij} := g_2(X_i, X_j)$, $0 \leqslant i, j \leqslant 2n-1$ are components of the metric g_2 with respect to the moving frame X_i. Taking into account equations $[X_0, X_i] = 0$, $\forall i = 0, \dots, 2n-1$, one gets:

$$\left(\pounds_{X_0} g_2\right)(X_i, X_j) = X_0 g_2(X_i, X_j) = X_0\left(\hat{g}_{2,ij}\right)\ .$$

Thus, using formula (2.17), we get the following additional term in the formula for the Laplace-Beltrami operator:

$$\begin{aligned}\frac{1}{2}X_0\left(\hat{g}_{2,kj}\right)\hat{g}_2^{kj}\hat{g}_2^{0i}X_i - X_0\left(\hat{g}_{2,kj}\right)\hat{g}_2^{0k}\hat{g}_2^{ji}X_i &= \frac{1}{2\hat{\gamma}}X_0(\hat{\gamma})\hat{g}_2^{0i}X_i + X_0(\hat{g}^{0i})X_i \\ &= \frac{1}{\sqrt{\hat{\gamma}}}X_0\left(\sqrt{\hat{\gamma}}\hat{g}_2^{0i}\right)X_i,\end{aligned}$$

where $\hat{\gamma} = \det \hat{g}_{2,ij}$. Finally, one has:

$$\triangle_{g_2} = \hat{g}^{ij} X_i \circ X_j + c_{jq}^q \hat{g}^{ji} X_i + \frac{1}{\sqrt{\hat{\gamma}}} X_0\left(\sqrt{\hat{\gamma}}\hat{g}_2^{0i}\right) X_i\ .$$

The vector field X_0 on the space $I' \times (G/K_0)$ has the form $\partial/\partial r$ and its lift onto the space $I' \times G$ is again $\partial/\partial r$. According to the Remark 2.4 and Lemma 2.5 one gets the following expression for the lift of the operator $\triangle_{g_2}$:

$$\tilde{\triangle}_{g_2} = \hat{g}^{ij}\big|_{x_0} X_i^l \circ X_j^l + \left(c_{jq}^q \hat{g}^{ji}\right)\big|_{x_0} X_i^l + \left[\frac{1}{\sqrt{\hat{\gamma}}} X_0^l \left(\sqrt{\hat{\gamma}} \hat{g}_2^{0i}\right)\right]\Big|_{x_0} X_i^l, \quad (5.3)$$

where $X_0^l := \partial/\partial r$.

The G-invariant measure μ_2 on the space $I' \times (G/K_0)$ has the form $\nu \otimes \mu$, where $\nu = \phi(r)dr$ is the measure on the interval I', and μ is a G-invariant measure on the space G/K_0. The measure on the space $I' \times G$, corresponding to μ_2, has the form $\tilde{\mu}_2 = \nu \otimes \mu_G$, where μ_G is the left-invariant measure on the group G, appropriately normalized.

Similarly to Sect. 2.1, one can define the bijection ζ between the set of functions on the space $I' \times (G/K_0)$ and the set of functions on the space $I' \times G$ that are invariant with respect to the right K_0-shifts. Denote by $\mathcal{L}^2\left(I' \times G, K_0, \tilde{\mu}_2\right)$ the Hilbert space of square integrable K_0-invariant functions on $I' \times G$ with respect to the measure $\tilde{\mu}_2$ and the right K_0-shifts. Thus, one gets the following isometry of Hilbert spaces:

$$\zeta : \mathcal{L}^2\left(M, \mu_2\right) \to \mathcal{L}^2\left(I' \times G, K_0, \tilde{\mu}_2\right),$$

and also $\tilde{\triangle}_{g_2} \circ \zeta = \zeta \circ \triangle_{g_2}$.

5.2 Two-Body Hamiltonian on a Compact Two-Point Homogeneous Space

In this section we shall find the concrete expression for the two-point Hamiltonian of the form (5.3) on an arbitrary compact two-point homogeneous space Q. Let

$$L, X_{\lambda,i}, Y_{\lambda,i}, X_{2\lambda,j}, Y_{2\lambda,j},\ i = 1, \ldots, q_1, j = 1, \ldots, q_2 \quad (5.4)$$

be the Killing vector fields on the space Q, corresponding to the elements of the algebra $\mathfrak{g}$ from the Proposition 1.4

$$\Lambda, e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,j}, f_{2\lambda,j},\ i = 1, \ldots, q_1, j = 1, \ldots, q_2 \quad (5.5)$$

with respect to the left action of the group G on the space Q. Due to (2.1) this correspondence changes signs of commutators. For example it holds $[L, X_{\lambda,i}] = \frac{1}{2}Y_{\lambda,i}$, $[L, Y_{\lambda,i}] = -\frac{1}{2}X_{\lambda,i}$ and so on. Define the curve $\hat{\gamma}$ on the space Q by the formula $\hat{\gamma}(s) = \exp\left(\frac{s}{R}\Lambda\right) x_0$. This curve coincides with the geodesic $\widetilde{\gamma}$ from the previous section according to the second claim of the following proposition.

Proposition 5.1. *1. Among all possible pairwise scalar products of fields (5.4) on the curve $\hat{\gamma}$ only products from the followings list are nonzero:*

$$g(L, L)|_{\hat{\gamma}} = R^2, \quad (5.6)$$

$$g(X_{\lambda,i}, X_{\lambda,i})|_{\hat{\gamma}} = \frac{R^2}{2}\left(1 + \cos\frac{s}{R}\right), \; i = 1, \ldots, q_1 , \tag{5.7}$$

$$g(X_{\lambda,i}, Y_{\lambda,i})|_{\hat{\gamma}} = -\frac{R^2}{2}\sin\frac{s}{R}, \; i = 1, \ldots, q_1 , \tag{5.8}$$

$$g(Y_{\lambda,i}, Y_{\lambda,i})|_{\hat{\gamma}} = \frac{R^2}{2}\left(1 - \cos\frac{s}{R}\right), \; i = 1, \ldots, q_1 , \tag{5.9}$$

$$g(X_{2\lambda,i}, X_{2\lambda,i})|_{\hat{\gamma}} = \frac{R^2}{2}\left(1 + \cos\frac{2s}{R}\right), \; i = 1, \ldots, q_2 , \tag{5.10}$$

$$g(X_{2\lambda,i}, Y_{2\lambda,i})|_{\hat{\gamma}} = -\frac{R^2}{2}\sin\frac{2s}{R}, \; i = 1, \ldots, q_2 , \tag{5.11}$$

$$g(Y_{2\lambda,i}, Y_{2\lambda,i})|_{\hat{\gamma}} = \frac{R^2}{2}\left(1 - \cos\frac{2s}{R}\right), \; i = 1, \ldots, q_2 ; \tag{5.12}$$

2. $\hat{\gamma}(s) = \widetilde{\gamma}(s), \; s \in [0, \operatorname{diam} Q)$.

Proof. By construction, the vector field L/R is tangent to the curve $\hat{\gamma}(s)$. Since

$$\frac{d}{ds} g(L, L)\bigg|_{\hat{\gamma}(s)} = \frac{2}{R} g([L, L], L) = 0 ,$$

one has

$$g\left(\frac{1}{R}L, \frac{1}{R}L\right)\bigg|_{\hat{\gamma}(s)} \equiv g\left(\frac{1}{R}L, \frac{1}{R}L\right)\bigg|_{\hat{\gamma}(0)} = \left\langle \frac{1}{R}\Lambda, \frac{1}{R}\Lambda \right\rangle = 1 ,$$

which is equivalent to (5.6). Therefore, the parameter s is the natural parameter on the curve $\hat{\gamma}$. Using the equality

$$\frac{d}{ds} g(X, Y)\bigg|_{\hat{\gamma}(s)} = \frac{L}{R}\left(g(X, Y)\right)\bigg|_{\hat{\gamma}(s)} = \frac{1}{R}(g([L, X], Y)\bigg|_{\hat{\gamma}(s)} + \frac{1}{R}(g(X, [L, Y])\bigg|_{\hat{\gamma}(s)}$$

for smooth vector fields X, Y on the curve $\hat{\gamma}$, the relations (1.6) and the connection of the metric $g(\cdot, \cdot)|_{T_{x_0}Q}$ with the scalar product $\langle \cdot, \cdot \rangle$ on the algebra $\mathfrak{g}$, one gets the system of linear differential equations and initial conditions for all possible pairwise scalar products of the fields (5.4) on the curve $\hat{\gamma}$. This system decomposes into a set of easily solvable subsystems. For example, one has

$$\frac{d}{ds} g(X_{\lambda,i}, X_{\lambda,i})\bigg|_{\hat{\gamma}(s)} = \frac{2}{R}\, g\left([L, X_{\lambda,i}], X_{\lambda,i}\right)|_{\hat{\gamma}(s)} = \frac{1}{R}\, g(Y_{\lambda,i}, X_{\lambda,i})|_{\hat{\gamma}(s)} ,$$

$$\begin{aligned}\frac{d}{ds} g(Y_{\lambda,i}, X_{\lambda,i})\bigg|_{\hat{\gamma}(s)} &= \frac{1}{R}\, g\left([L, Y_{\lambda,i}], X_{\lambda,i}\right)|_{\hat{\gamma}(s)} + \frac{1}{R}\, g\left(Y_{\lambda,i}, [L, X_{\lambda,i}]\right)|_{\hat{\gamma}(s)} \\ &= -\frac{1}{2R}\, g(X_{\lambda,i}, X_{\lambda,i})|_{\hat{\gamma}(s)} + \frac{1}{2R}\, g(Y_{\lambda,i}, Y_{\lambda,i})|_{\hat{\gamma}(s)} ,\end{aligned}$$

$$\frac{d}{ds} g(Y_{\lambda,i}, Y_{\lambda,i})\Big|_{\hat{\gamma}(s)} = \frac{2}{R}\, g\left([L, Y_{\lambda,i}], Y_{\lambda,i}\right)|_{\hat{\gamma}(s)} = -\frac{1}{R}\, g(X_{\lambda,i}, Y_{\lambda,i})|_{\hat{\gamma}(s)} \ .$$

Taking into account the initial conditions given by

$$g(X_{\lambda,i}, X_{\lambda,i})|_{\hat{\gamma}(0)} = \langle e_{\lambda,i}, e_{\lambda,i}\rangle = R^2,$$
$$g(X_{\lambda,i}, Y_{\lambda,i})|_{\hat{\gamma}(0)} = g(Y_{\lambda,i}, Y_{\lambda,i})|_{\hat{\gamma}(0)} = 0, i = 1, \dots, q_1 \ ,$$

(valid due to the formula $Y_{\lambda,i}|_{\hat{\gamma}(0)} = 0$), one gets (5.7)–(5.9). Other formulae of the first statement can be derived in a similar way.

Let us prove the equality $\hat{\gamma}(s) = \widetilde{\gamma}(s)$. It is sufficient to show that $\nabla_L L|_{\hat{\gamma}(s)} = 0$, since the parameters of the curves $\hat{\gamma}(s), \widetilde{\gamma}(s)$ are natural. Formulae (2.21), (1.6) and the first statement of the Proposition 5.1 imply

$$\begin{aligned} g(\nabla_L L, X_{\lambda,i})|_{\hat{\gamma}(s)} &= g(L, [L, X_{\lambda,i}])|_{\hat{\gamma}(s)} = \frac{1}{2}\, g(L, Y_{\lambda,i})|_{\hat{\gamma}(s)} = 0,\ i = 1, \dots, q_1, \\ g(\nabla_L L, X_{2\lambda,j})|_{\hat{\gamma}(s)} &= g(L, [L, X_{2\lambda,j}])|_{\hat{\gamma}(s)} = g(L, Y_{2\lambda,j})|_{\hat{\gamma}(s)} = 0,\ j = 1, \dots, q_2, \\ g(\nabla_L L, L)|_{\hat{\gamma}(s)} &= g(L, [L, L])|_{\hat{\gamma}(s)} = 0 \ . \end{aligned} \tag{5.13}$$

Due to the first statement of this proposition the vector fields

$$L, X_{\lambda,i}, X_{2\lambda,j},\ i = 1, \dots, q_1, j = 1, \dots, q_2$$

form a moving frame in the tangent spaces $T_{\hat{\gamma}(s)}Q$ as $s \in [0, \operatorname{diam} Q)$, since the matrix of their pairwise scalar products in these spaces is nonsingular. Thus, due to (5.13), one has $\nabla_L L|_{\hat{\gamma}(s)} \equiv 0,\ s \in [0, \operatorname{diam} Q]$. □

Let $K \supset K_0$ be a subgroup of the group G, conserving the point $\mathbf{x}_0 = \widetilde{\gamma}(0)$, and $\mathfrak{k}$ be its Lie algebra. The two-point homogeneity of the space Q implies that K acts transitively on a subset

$$Q_p := (\mathbf{x} \in Q|\ \rho(\mathbf{x}, \mathbf{x}_0) = p = \text{const}),\ 0 \leqslant p \leqslant \operatorname{diam} Q$$

of Q. Due to Proposition 1.1 the stationary subgroup of this action, corresponding to the point $\widetilde{\gamma} \cap Q_p$ for $0 < p < \operatorname{diam} Q$, is the group K_0. This consideration leads to the following lemma, which will be used later.

Lemma 5.2. *The subspace*

$$Q' := (\mathbf{x} \in Q|\ 0 < \rho(\mathbf{x}, \mathbf{x}_0) < \operatorname{diam} Q)$$

is the following direct product

$$Q' = \bigcup_{s \in (0, \operatorname{diam} Q)} K\widetilde{\gamma}(s) = I \times (K/K_0),$$

where $I = (0, \operatorname{diam} Q)$.

Remark 5.1. *It was mentioned above in Sect.* 1.2 *that the decomposition* $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ *is uniquely determined by the point* x_0*. Therefore, due to proposition* 5.1 *and the isotropy of the space* Q *all nonzero elements of the space* $\mathfrak{p}$ *have the following property: the trajectories of all one-parameter subgroups corresponding to these elements and passing through the point* x_0 *are geodesics. In particular it holds for the elements* $e_{\lambda,i}, e_{2\lambda,j}$, $i = 1, \dots, q_1, j = 1, \dots, q_2$.

Let us rename some notations to simplify the consideration of the space $M = Q \times Q$. Now, let

$$L = L^{(1)} + L^{(2)},\ X_{\lambda,i} = X^{(1)}_{\lambda,i} + X^{(2)}_{\lambda,i},\ Y_{\lambda,i} = Y^{(1)}_{\lambda,i} + Y^{(2)}_{\lambda,i},\ i = 1, \dots, q_1,$$
$$X_{2\lambda,j} = X^{(1)}_{2\lambda,j} + X^{(2)}_{2\lambda,j},\ Y_{2\lambda,j} = Y^{(1)}_{2\lambda,j} + Y^{(2)}_{2\lambda,j},\ j = 1, \dots, q_2$$

be the decomposition of Killing vector fields on the space M which correspond to the elements $\Lambda, e_{\lambda,i}, f_{\lambda,i}, e_{2\lambda,j}, f_{2\lambda,j}$ and the decomposition $T_{(x_1,x_2)}M = T_{x_1}Q \oplus T_{x_2}Q$. Let $\gamma(p)$ be a curve on the space M, constructed according to Sect. 5.1 with respect to the geodesic $\widetilde{\gamma}$, and X_0 be the vector field on the space M constructed therein. Put $s_1(p) = \alpha p,\ s_2(p) = -\beta p,\ \alpha, \beta \in (0, 1),\ \alpha + \beta = 1,\ p =: 2R \arctan r,\ r \in I'$, where $I' = (0, \infty)$ for $Q \neq \mathbf{P}^n(\mathbb{R})$ and $I' = (0, 1)$ for $Q = \mathbf{P}^n(\mathbb{R})$. Then

$$X_0 = \frac{d}{dr}\gamma(p(r)) = \frac{2}{1 + r^2}\left(\alpha L^{(1)} - \beta L^{(2)}\right) \tag{5.14}$$

and $\widetilde{\gamma}(s)$ is the normal parametrization of $\widetilde{\gamma}$. Let us show that the vector fields

$$X_0,\ L,\ X_{\lambda,i},\ Y_{\lambda,i},\ X_{2\lambda,j},\ Y_{2\lambda,j},\ i = 1, \dots, q_1, j = 1, \dots, q_2 \tag{5.15}$$

form a moving frame in a neighborhood of the curve $\gamma(p),\ p \in (0, \operatorname{diam} Q)$. To prove this, we shall find the matrix $\mathcal{B}$ of pairwise scalar products of these fields on the curve γ. Since $(\tilde{\pi}^*_k g)(L, L) = R^2,\ k = 1, 2$, one has

$$\begin{aligned}
g_2(X_0, X_0)|_\gamma &= g_2\left(\frac{2}{1 + r^2}\left(\alpha L^{(1)} - \beta L^{(2)}\right), \frac{2}{1 + r^2}\left(\alpha L^{(1)} - \beta L^{(2)}\right)\right)\Big|_\gamma \\
&= \frac{4R^2}{(1 + r^2)^2}(\alpha^2 m_1 + \beta^2 m_2) =: a, \\
g_2(L, X_0)|_\gamma &= g_2\left(L^{(1)} + L^{(2)}, \frac{2}{1 + r^2}\left(\alpha L^{(1)} - \beta L^{(2)}\right)\right)\Big|_\gamma \\
&= \frac{2R^2}{1 + r^2}(\alpha m_1 - \beta m_2) =: b, \\
g_2(L, L)|_\gamma &= (m_1 + m_2)R^2 =: c\,.
\end{aligned}$$

Due to (5.14) and the orthogonality of the fields $L^{(k)},\ k = 1, 2$ with respect to all fields

$$X^{(k)}_{\lambda,i},\ Y^{(k)}_{\lambda,i},\ X^{(k)}_{2\lambda,j},\ Y^{(k)}_{2\lambda,j},\ i = 1, \dots, q_1, j = 1, \dots, q_2, k = 1, 2$$

one gets the orthogonality of the vector fields X_0, L with respect to the fields

$$X_{\lambda,i},\ Y_{\lambda,i},\ X_{2\lambda,j},\ Y_{2\lambda,j},\ i=1,\dots,q_1, j=1,\dots,q_2\ . \tag{5.16}$$

Proposition 5.1 implies that all possible pairwise scalar products of the fields (5.16) vanish except of $(X_{\lambda,i},Y_{\lambda,i}),(X_{2\lambda,j},Y_{2\lambda,j}),\ i=1,\dots,q_1,\ j=1,\dots,q_2$ and scalar squares. By simple calculations, taking into account (5.7)–(5.12), one obtains

$$\begin{aligned}
g_2(X_{\lambda,i},X_{\lambda,i})\big|_\gamma &= R^2\left(m_1\cos^2(\alpha\arctan r)+m_2\cos^2(\beta\arctan r)\right) =: d,\\
g_2(X_{\lambda,i},Y_{\lambda,i})\big|_\gamma &= R^2\left(-m_1\sin(\alpha\arctan r)\cos(\alpha\arctan r)\right.\\
&\quad \left.+m_2\sin(\beta\arctan r)\cos(\beta\arctan r)\right) =: h,\\
g_2(Y_{\lambda,i},Y_{\lambda,i})\big|_\gamma &= R^2\left(m_1\sin^2(\alpha\arctan r)+m_2\sin^2(\beta\arctan r)\right) =: f,\\
g_2(X_{2\lambda,j},X_{2\lambda,j})\big|_\gamma &= R^2\left(m_1\cos^2(2\alpha\arctan r)+m_2\cos^2(2\beta\arctan r)\right) =: u,\\
g_2(X_{2\lambda,j},Y_{2\lambda,j})\big|_\gamma &= R^2\left(-m_1\sin(2\alpha\arctan r)\cos(2\alpha\arctan r)\right.\\
&\quad \left.+m_2\sin(2\beta\arctan r)\cos(2\beta\arctan r)\right) =: w,\\
g_2(Y_{2\lambda,j},Y_{2\lambda,j})\big|_\gamma &= R^2\left(m_1\sin^2(2\alpha\arctan r)+m_2\sin^2(2\beta\arctan r)\right) =: v,\\
&\quad i=1,\dots,q_1, j=1,\dots,q_2\ .
\end{aligned}$$

Thus, one concludes that the matrix $\mathcal{B}=g_2|_\gamma$ has a block structure with the following blocks:

$$\begin{pmatrix} a & b\\ b & c\end{pmatrix}\ \text{one time,}\ \begin{pmatrix} d & h\\ h & f\end{pmatrix}\ -q_1\ \text{times and}\ \begin{pmatrix} u & w\\ w & v\end{pmatrix}\ -q_2\ \text{times.}$$

Therefore, it holds $\det\mathcal{B}=(ac-b^2)(df-h^2)^{q_1}(uv-w^2)^{q_2}$. It is easy to show that

$$ac-b^2=\frac{4R^4m_1m_2}{(1+r^2)^2},\ df-h^2=\frac{R^4m_1m_2r^2}{1+r^2},\ uv-w^2=\frac{4R^4m_1m_2r^2}{(1+r^2)^2}\ .$$

Thus, one gets

$$\det\mathcal{B}=\frac{4^{1+q_2}\left(R^4m_1m_2\right)^{1+q_1+q_2}r^{2(q_1+q_2)}}{(1+r^2)^{2+q_1+2q_2}}, \tag{5.17}$$

$$\begin{pmatrix} a & b\\ b & c\end{pmatrix}^{-1}=\frac{1}{4R^2m_1m_2}\begin{pmatrix}(1+r^2)^2(m_1+m_2) & -2(1+r^2)(m_1\alpha-m_2\beta)\\ -2(1+r^2)(m_1\alpha-m_2\beta) & 4(m_1\alpha^2+m_2\beta^2)\end{pmatrix},$$
$$\begin{pmatrix} d & h\\ h & f\end{pmatrix}^{-1}=\begin{pmatrix} D_s & E_s\\ E_s & F_s\end{pmatrix},\ \begin{pmatrix} u & w\\ w & v\end{pmatrix}^{-1}=\begin{pmatrix} C_s & B_s\\ B_s & A_s\end{pmatrix},\ \text{where}$$

$$\begin{aligned}
D_s &= \frac{1+r^2}{m_1m_2R^2r^2}\left(m_1\sin^2(\alpha\arctan r)+m_2\sin^2(\beta\arctan r)\right),\\
F_s &= \frac{1+r^2}{m_1m_2R^2r^2}\left(m_1\cos^2(\alpha\arctan r)+m_2\cos^2(\beta\arctan r)\right),
\end{aligned}$$

$$E_s = \frac{1+r^2}{2m_1m_2R^2r^2}\left(m_1\sin(2\alpha\arctan r) - m_2\sin(2\beta\arctan r)\right),$$
$$C_s = \frac{(1+r^2)^2}{4m_1m_2R^2r^2}\left(m_1\sin^2(2\alpha\arctan r) + m_2\sin^2(2\beta\arctan r)\right),$$
$$A_s = \frac{(1+r^2)^2}{4m_1m_2R^2r^2}\left(m_1\cos^2(2\alpha\arctan r) + m_2\cos^2(2\beta\arctan r)\right),$$
$$B_s = \frac{(1+r^2)^2}{8m_1m_2R^2r^2}\left(m_1\sin(4\alpha\arctan r) - m_2\sin(4\beta\arctan r)\right)\ .$$

Due to (5.17) vector fields (5.15) form a moving frame on the curve $\gamma(p)$, $p \in (0, \operatorname{diam} Q)$ and condition 5.1 is satisfied. Let

$$L^l, X^l_{\lambda,i}, Y^l_{\lambda,i}, X^l_{2\lambda,j}, Y^l_{2\lambda,j},\ i = 1,\dots,q_1, j = 1,\dots,q_2 \tag{5.18}$$

be left-invariant vector fields on the group G, corresponding to elements (5.5) of the algebra $\mathfrak{g}$, and $X^l_0 = \partial/\partial r$ be a vector field on I'. We consider the corresponding fields on the space $I' \times G$ saving the notations. The field X_0 commutes with all fields (5.15). So, due to the Proposition 1.4, the expansion of the commutator $[X, Y]$ for X, Y being elements of the frame (5.15), by the same frame, does not include X, Y. Thus, the second term in the lift $\tilde{H}_0$ of the two-body Hamiltonian H_0 onto the space $I' \times G$ in accordance with (5.3) vanishes, since $c^q_{jq} = 0$ (even without summation over q). Consequently, this expression has the form:

$$\begin{aligned}
\tilde{H}_0 = &-\frac{(1+r^2)^{1+\frac{q_1}{2}+q_2}}{8mR^2r^{q_1+q_2}}\frac{\partial}{\partial r}\circ\left(\frac{r^{q_1+q_2}}{(1+r^2)^{\frac{q_1}{2}+q_2-1}}\frac{\partial}{\partial r}\right)\\
&+\frac{(m_1\alpha - m_2\beta)(1+r^2)^{1+\frac{q_1}{2}+q_2}}{4m_1m_2R^2r^{q_1+q_2}}\times\left\{\frac{\partial}{\partial r}, \frac{r^{q_1+q_2}}{(1+r^2)^{\frac{q_1}{2}+q_2}}L^l\right\}\\
&-\frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}\left(L^l\right)^2 - \frac{1}{2}\sum_{i=1}^{q_1}\left(D_s\left(X^l_{\lambda,i}\right)^2 + F_s\left(Y^l_{\lambda,i}\right)^2 + E_s\left\{X^l_{\lambda,i}, Y^l_{\lambda,i}\right\}\right)\\
&-\frac{1}{2}\sum_{j=1}^{q_2}\left(C_s\left(X^l_{2\lambda,j}\right)^2 + A_s\left(Y^l_{2\lambda,j}\right)^2 + B_s\left\{X^l_{2\lambda,j}, Y^l_{2\lambda,j}\right\}\right)\ ,
\end{aligned}\tag{5.19}$$

where $\{X, Y\} = X \circ Y + Y \circ X$ is the anticommutator of X and Y, and $m := \dfrac{m_1m_2}{m_1+m_2}$.

According to Sect. 5.1, the lift of the measure, generated by the metric g_2, onto the space $I' \times G$ has the form $\tilde{\mu}_2 = \nu \otimes \mu_G$, where $\nu = \sqrt{\det \mathcal{B}}dr$ is the measure on I', and μ_G is the biinvariant measure on the group G. Changing the normalization one gets $\nu = r^{q_1+q_2}dr/(1+r^2)^{1+\frac{q_1}{2}+q_2}$. The calculations above can be summarized in the following theorem.

Theorem 5.1. *The quantum two-body Hamiltonian on a compact two-point homogeneous space Q with the connected isometry group G can be considered as the differential operator $\tilde{H}_0 + V(r)$, where the operator $\tilde{H}_0$ on the space $I' \times G$ is given by formula* (5.19), *$I' = (0, 1)$ in the case $Q = \mathbf{P}^n(\mathbb{R})$ and*

$I' = (0, \infty)$ in other cases, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$. Its domain is dense in the space $\mathcal{L}^2\left(I' \times G, K_0, \tilde{\mu}_2\right)$, consisting of all square integrable K_0-invariant functions on $I' \times G$, with respect to the measure $\tilde{\mu}_2$ and right K_0-shifts.

5.3 Two-Body Hamiltonian on a Noncompact Two-Point Homogeneous Space

Noncompact two-point homogeneous spaces of types 7,8,9,10 are analogous to the compact two-point homogeneous spaces of types 2,4,5,6, respectively. According to Proposition 1.5 Lie algebras of isometry groups of analogous spaces are different real forms of a simple complex Lie algebra. The transition from one such real form to another can be done by multiplying the subspace $\mathfrak{p}$ from the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ by the imaginary unit $\mathbf{i}$. In the space $M = Q \times Q$ this transition corresponds to the change $r \to -\mathbf{i}r$, $R \to \mathbf{i}R$, see Sect. 1.3.3.

Thus, changing variables and vector fields in (5.19) as

$$\begin{aligned} &r \to -\mathbf{i}r,\ R \to \mathbf{i}R,\ X^l_{\lambda,i} \to \mathbf{i}X^l_{\lambda,i},\ X^l_{2\lambda,j} \to \mathbf{i}X^l_{2\lambda,j},\\ &L^l \to \mathbf{i}L^l,\ Y^l_{\lambda,i} \to \mathbf{i}Y^l_{\lambda,i},\ Y^l_{2\lambda,j} \to \mathbf{i}Y^l_{2\lambda,j}, \end{aligned}$$

and using (1.30) one gets

Theorem 5.2. *The quantum two-body Hamiltonian on a noncompact two-point homogeneous space Q with the connected isometry group G can be considered as the differential operator*

$$\begin{aligned} \tilde{H} = &-\frac{(1-r^2)^{1+\frac{q_1}{2}+q_2}}{8mR^2 r^{q_1+q_2}}\frac{\partial}{\partial r}\circ\left(\frac{r^{q_1+q_2}}{(1-r^2)^{\frac{q_1}{2}+q_2-1}}\frac{\partial}{\partial r}\right)\\ &+\frac{(m_1\alpha - m_2\beta)(1-r^2)^{1+\frac{q_1}{2}+q_2}}{4m_1m_2R^2r^{q_1+q_2}}\left\{\frac{\partial}{\partial r}, \frac{r^{q_1+q_2}}{(1-r^2)^{\frac{q_1}{2}+q_2}}L^l\right\}\\ &-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}\left(L^l\right)^2 - \frac{1}{2}\sum_{i=1}^{q_1}\Big(D_h\left(X^l_{\lambda,i}\right)^2 + F_h\left(Y^l_{\lambda,i}\right)^2 + E_h\left\{X^l_{\lambda,i}, Y^l_{\lambda,i}\right\}\Big)\\ &-\frac{1}{2}\sum_{j=1}^{q_2}\Big(C_h\left(X^l_{2\lambda,j}\right)^2 + A_h\left(Y^l_{2\lambda,j}\right)^2 + B_h\left\{X^l_{2\lambda,j}, Y^l_{2\lambda,j}\right\}\Big) + V(r)\,, \quad (5.20) \end{aligned}$$

on the space $I' \times G$, where $I' = (0,1)$,

$$\begin{aligned} D_h &= \frac{1-r^2}{m_1m_2R^2r^2}\left(m_1\sinh^2(\alpha\operatorname{arctanh} r) + m_2\sinh^2(\beta\operatorname{arctanh} r)\right),\\ F_h &= \frac{1-r^2}{m_1m_2R^2r^2}\left(m_1\cosh^2(\alpha\operatorname{arctanh} r) + m_2\cosh^2(\beta\operatorname{arctanh} r)\right),\\ E_h &= \frac{1-r^2}{2m_1m_2R^2r^2}\left(m_1\sinh(2\alpha\operatorname{arctanh} r) - m_2\sinh(2\beta\operatorname{arctanh} r)\right),\\ C_h &= \frac{(1-r^2)^2}{4m_1m_2R^2r^2}\left(m_1\sinh^2(2\alpha\operatorname{arctanh} r) + m_2\sinh^2(2\beta\operatorname{arctanh} r)\right), \end{aligned}$$

$$A_h = \frac{(1-r^2)^2}{4m_1m_2R^2r^2}\left(m_1\cosh^2(2\alpha\,\mathrm{arctanh}\,r) + m_2\cosh^2(2\beta\,\mathrm{arctanh}\,r)\right),$$
$$B_h = \frac{(1-r^2)^2}{8m_1m_2R^2r^2}\left(m_1\sinh(4\alpha\,\mathrm{arctanh}\,r) - m_2\sinh(4\beta\,\mathrm{arctanh}\,r)\right)\ . \quad (5.21)$$

Its domain is dense in the space $\mathcal{L}^2\left(I'\times G, K_0, \tilde{\mu}_2\right)$, *consisting of all square-integrable* K_0*-invariant functions on* $I'\times G$, *with respect to the measure* $\tilde{\mu}_2 = \nu\otimes\mu_G$ *and the right* K_0*-shifts. Now*

$$\nu = \frac{r^{q_1+q_2}dr}{(1-r^2)^{1+\frac{q_1}{2}+q_2}}$$

is the measure on I' *and* μ_G *is biinvariant measure on* G, *since* G *is unimodular.*

The following remark is analogous to Remarks 5.1 and 1.1.

Remark 5.2. *The space* $\mathfrak{a}\oplus\mathfrak{p}_{2\lambda}$ *generates in the space* Q *a completely geodesic submanifold of the constant sectional curvature* $-R^{-2}$, *isometric to the space* $\mathbf{H}^{q_2+1}(\mathbb{R})$.

If $q_1 \neq 0$, *the element* Λ *and an arbitrary nonzero element from the space* $\mathfrak{p}_\lambda$ *generate in* Q *a completely geodesic two-dimensional submanifolds of the constant curvature* $-(2R)^{-2}$.

Trajectories of all one-parameter subgroups corresponding to elements of the space $\mathfrak{p}$, *passing through the point* x_0, *are geodesics. In particular, it holds for the elements* $e_{\lambda,i}, e_{2\lambda,j}$, $i=1,\ldots,q_1, j=1,\ldots,q_2$.

5.4 Connection of the Two-Body Hamiltonian and the Algebra $\mathrm{Diff}_G(Q_{\mathbf{S}})$

Now one can express the quantum two-body Hamiltonian with a central potential $V(\rho)$ on an arbitrary two-point homogeneous space Q through radial differential operators and generators of algebras of invariant differential operators on the unit sphere bundle over Q. This generators were calculated in Chap. 3, since $Q_S \cong G/K_0$. Comparing expression (5.19) with generators of these algebras one gets

$$\begin{aligned} H = &-\frac{(1+r^2)^{1+\frac{q_1}{2}+q_2}}{8mR^2r^{q_1+q_2}}\frac{\partial}{\partial r}\circ\left(\frac{r^{q_1+q_2}}{(1+r^2)^{\frac{q_1}{2}+q_2-1}}\frac{\partial}{\partial r}\right) - \frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}D_0^2 \\ &+ \frac{(m_1\alpha - m_2\beta)(1+r^2)^{1+\frac{q_1}{2}+q_2}}{4m_1m_2R^2r^{q_1+q_2}}\left\{\frac{\partial}{\partial r}, \frac{r^{q_1+q_2}D_0}{(1+r^2)^{\frac{q_1}{2}+q_2}}\right\} \\ &- \frac{1}{2}\left(D_sD_1 + F_sD_2 + 2E_sD_3 + C_sD_4 + A_sD_5 + 2B_sD_6\right) + V(r)\ , \end{aligned} \quad (5.22)$$

for $Q = \mathbf{P}^n(\mathbb{H})$, $q_1 = 4n-4, q_2 = 3$ and $Q = \mathbf{P}^2(\mathbb{C}a)$, $q_1 = 8, q_2 = 7$;

$$H = -\frac{(1+r^2)^{n+1}}{8mR^2r^{2n-1}}\frac{\partial}{\partial r}\circ\left(\frac{r^{2n-1}}{(1+r^2)^{n-1}}\frac{\partial}{\partial r}\right) - \frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}\left(D_0\right)^2$$

$$+\frac{(m_1\alpha-m_2\beta)(1+r^2)^{n+1}}{4m_1m_2R^2r^{2n-1}}\left\{\frac{\partial}{\partial r},\frac{r^{2n-1}}{(1+r^2)^n}D_0\right\} \tag{5.23}$$
$$-\frac{1}{2}\left(D_sD_1+F_sD_2+2E_sD_3+C_sD_4^2+A_sD_5^2+B_s\{D_4,D_5\}\right)+V(r)\ ,$$

for $Q=\mathbf{P}^n(\mathbb{C})$;

$$H=-\frac{(1+r^2)^n}{8mR^2r^{n-1}}\frac{\partial}{\partial r}\circ\left(\frac{r^{n-1}}{(1+r^2)^{n-2}}\frac{\partial}{\partial r}\right)-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}D_0^2 \tag{5.24}$$
$$+\frac{(m_1\alpha-m_2\beta)(1+r^2)^n}{4m_1m_2R^2r^{n-1}}\left\{\frac{\partial}{\partial r},\frac{r^{n-1}D_0}{(1+r^2)^{n-1}}\right\}$$
$$-\frac{1}{2}\left(C_sD_1+A_sD_2+2B_sD_3\right)+V(r)\ ,$$

for $Q=\mathbf{P}^n(\mathbb{R})$, $\mathbf{S}^n, n\geqslant 3$ and

$$H=-\frac{(1+r^2)^2}{8mR^2r}\frac{\partial}{\partial r}\circ\left(r\frac{\partial}{\partial r}\right)+\frac{(m_1\alpha-m_2\beta)(1+r^2)^2}{4m_1m_2R^2r}\left\{\frac{\partial}{\partial r},\frac{rD_0}{1+r^2}\right\}$$
$$-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}D_0^2-\frac{1}{2}\left(C_sD_1^2+A_sD_2^2+B_s\{D_1,D_2\}\right)+V(r), \tag{5.25}$$

for $Q=\mathbf{P}^2(\mathbb{R})$, $\mathbf{S}^2$.

The analogous expressions for noncompact spaces can be obtained by the substitution $r\to-\mathbf{i}r$, $R\to\mathbf{i}R$, $D_i\to c_i\bar{D}_i$, where $c_i=\pm1,\pm\mathbf{i}$ (see Chap. 3), taking into account that

$$D_s\to-D_h,\ F_s\to F_h,\ E_s\to-\mathbf{i}E_h,\ C_s\to-C_h,\ A_s\to A_h,\ B_s\to-\mathbf{i}B_h\ .$$

They are

$$H=-\frac{(1-r^2)^{1+\frac{q_1}{2}+q_2}}{8mR^2r^{q_1+q_2}}\frac{\partial}{\partial r}\circ\left(\frac{r^{q_1+q_2}}{(1-r^2)^{\frac{q_1}{2}+q_2-1}}\frac{\partial}{\partial r}\right)-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}\bar{D}_0^2$$
$$+\frac{(m_1\alpha-m_2\beta)(1-r^2)^{1+\frac{q_1}{2}+q_2}}{4m_1m_2R^2r^{q_1+q_2}}\left\{\frac{\partial}{\partial r},\frac{r^{q_1+q_2}\bar{D}_0}{(1-r^2)^{\frac{q_1}{2}+q_2}}\right\} \tag{5.26}$$
$$-\frac{1}{2}\left(D_h\bar{D}_1+F_h\bar{D}_2+2E_h\bar{D}_3+C_h\bar{D}_4+A_h\bar{D}_5+2B_h\bar{D}_6\right)+V(r),$$

for $Q=\mathbf{H}^n(\mathbb{H})$, $q_1=4n-4, q_2=3$ and $Q=\mathbf{H}^2(\mathbb{C}a)$, $q_1=8, q_2=7$;

$$H=-\frac{(1-r^2)^{n+1}}{8mR^2r^{2n-1}}\frac{\partial}{\partial r}\circ\left(\frac{r^{2n-1}}{(1-r^2)^{n-1}}\frac{\partial}{\partial r}\right)-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}\left(\bar{D}_0\right)^2$$
$$+\frac{(m_1\alpha-m_2\beta)(1-r^2)^{n+1}}{4m_1m_2R^2r^{2n-1}}\left\{\frac{\partial}{\partial r},\frac{r^{2n-1}}{(1-r^2)^n}\bar{D}_0\right\} \tag{5.27}$$
$$-\frac{1}{2}\left(D_h\bar{D}_1+F_h\bar{D}_2+2E_h\bar{D}_3+C_h\bar{D}_4^2+A_h\bar{D}_5^2+B_h\{\bar{D}_4,\bar{D}_5\}\right)+V(r),$$

for $Q=\mathbf{H}^n(\mathbb{C})$;

$$\begin{aligned} H = &-\frac{(1-r^2)^n}{8mR^2r^{n-1}}\frac{\partial}{\partial r}\circ\left(\frac{r^{n-1}}{(1-r^2)^{n-2}}\frac{\partial}{\partial r}\right)-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}\bar{D}_0^2 \\ &+\frac{(m_1\alpha-m_2\beta)(1-r^2)^n}{4m_1m_2R^2r^{n-1}}\left\{\frac{\partial}{\partial r},\frac{r^{n-1}\bar{D}_0}{(1-r^2)^{n-1}}\right\} \\ &-\frac{1}{2}\left(C_h\bar{D}_1+A_h\bar{D}_2+2B_h\bar{D}_3\right)+V(r)\,, \end{aligned} \tag{5.28}$$

for $Q=\mathbf{H}^n(\mathbb{R})$, $n\geqslant 3$ and

$$\begin{aligned} H = &-\frac{(1-r^2)^2}{8mR^2r}\frac{\partial}{\partial r}\circ\left(r\frac{\partial}{\partial r}\right)+\frac{(m_1\alpha-m_2\beta)(1-r^2)^2}{4m_1m_2R^2r}\left\{\frac{\partial}{\partial r},\frac{r\bar{D}_0}{1-r^2}\right\} \\ &-\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}\bar{D}_0^2-\frac{1}{2}\left(C_h\bar{D}_1^2+A_h\bar{D}_2^2+B_h\{\bar{D}_1,\bar{D}_2\}\right)+V(r)\,, \end{aligned} \tag{5.29}$$

for $Q=\mathbf{H}^2(\mathbb{R})$ (everywhere in (5.26)–(5.29) we suppose $0<r<1$).

The main difference of these expressions from Euclidean case is the presence of noncommutative operators with coefficients depending on r. This difference makes the two-body problem on two-point homogeneous spaces quite difficult. However, every common eigenfunction of generators D_i gives an isolated ordinary differential equation for a radial part of an eigenfunction for H. Using this approach some exact spectral series for the two-body problem on $\mathbf{S}^{\mathbf{n}}$ will be found for several potentials below. For other two-point compact homogeneous spaces similar calculations should be more difficult.

Remark 5.3. *Informally, the more independent operators commute with a Hamiltonian, the closer a quantum system is to integrability. Expressions* (5.22)–(5.29) *for two-body Hamiltonians seem to be complicated, nevertheless in the case of the trivial potential* $V\equiv 0$ *each of them evidently could be represented as the sum of two commuting operators* H_1 *and* H_2*, corresponding to independent particles. One of these operator is proportional to* $1/m_1$ *and another one to* $1/m_2$*. For instance the Hamiltonian* (5.26) *is* $H_1+H_2+V(r)$*, where*

$$\begin{aligned} H_1 = &-\frac{(1+r^2)^{1+\frac{q_1}{2}+q_2}}{8m_1R^2r^{q_1+q_2}}\frac{\partial}{\partial r}\circ\left(\frac{r^{q_1+q_2}}{(1+r^2)^{\frac{q_1}{2}+q_2-1}}\frac{\partial}{\partial r}\right)-\frac{\beta^2}{2m_1R^2}D_0^2 \\ &-\frac{\beta(1+r^2)^{1+\frac{q_1}{2}+q_2}}{4m_1R^2r^{q_1+q_2}}\left\{\frac{\partial}{\partial r},\frac{r^{q_1+q_2}D_0}{(1+r^2)^{\frac{q_1}{2}+q_2}}\right\} \\ &-\frac{1+r^2}{2m_1R^2r^2}\left(\sin^2(\beta\arctan r)D_1+\cos^2(\beta\arctan r)D_2-\sin(2\beta\arctan r)D_3\right) \\ &-\frac{(1+r^2)^2}{8m_1R^2r^2}\left(\sin^2(2\beta\arctan r)D_4+\cos^2(2\beta\arctan r)D_5-\sin(4\beta\arctan r)D_6\right), \\ H_2 = &-\frac{(1+r^2)^{1+\frac{q_1}{2}+q_2}}{8m_2R^2r^{q_1+q_2}}\frac{\partial}{\partial r}\circ\left(\frac{r^{q_1+q_2}}{(1+r^2)^{\frac{q_1}{2}+q_2-1}}\frac{\partial}{\partial r}\right)-\frac{\alpha^2}{2m_2R^2}D_0^2 \\ &+\frac{\alpha(1+r^2)^{1+\frac{q_1}{2}+q_2}}{4m_2R^2r^{q_1+q_2}}\left\{\frac{\partial}{\partial r},\frac{r^{q_1+q_2}D_0}{(1+r^2)^{\frac{q_1}{2}+q_2}}\right\} \end{aligned}$$

$$
\begin{aligned}
&- \frac{1+r^2}{2m_2R^2r^2}\Big(\sin^2(\alpha\arctan r)D_1 + \cos^2(\alpha\arctan r)D_2 + \sin(2\alpha\arctan r)D_3\Big) \\
&- \frac{(1+r^2)^2}{8m_2R^2r^2}\Big(\sin^2(2\alpha\arctan r)D_4 + \cos^2(2\alpha\arctan r)D_5 + \sin(4\alpha\arctan r)D_6\Big) \ .
\end{aligned}
$$

However, it is not clear if it is possible to incorporate a nontrivial potential into operators H_1, H_2 not disturbing their commutativity and in such a way that it would be $H = H_1 + H_2$.

6

Particle in a Central Field on Two-Point Homogeneous Spaces

In this chapter we consider the one-body problem in a central potential on two-point homogeneous Riemannian spaces. In Sect. 6.1 we prove its noncommutative integrability for such spaces different from constant curvature spaces.

The other part of this chapter deals with the classical and quantum one-body problems in central potentials in constant curvature spaces. There are many similarities between these problems and their Euclidean counterparts. Namely there are the Newton-Coulomb and oscillator potentials here, which imply the motion of a classical particles along conic trajectories. The solution of the Kepler problem in these spaces also satisfies analogs of the Kepler laws. The corresponding quantum problems are exactly solvable. These results are collected together for the first time.

The last section of the present chapter describes the history of these problems since the rise of the noneuclidean geometry.

6.1 Integrability of the One-Particle Motion in a Central Field on Two-Point Homogeneous Spaces

Here we shall prove the noncommutative integrability according to Definition 4.4 of a classical one-body motion in a central field on an arbitrary two-point homogeneous space Q. To do this it is enough to consider only the spaces $\mathbf{P}^2(\mathbb{C}a), \mathbf{P}^2(\mathbb{H}), \mathbf{P}^2(\mathbb{C}), \mathbf{P}^2(\mathbb{R}), \mathbf{S}^2$. Indeed, it is obvious from the consideration of the isometry group for spaces $\mathbf{P}^n(\mathbb{H}), \mathbf{P}^n(\mathbb{C}), \mathbf{P}^n(\mathbb{R}), \mathbf{S}^n$ that there is a totally geodesic subspace Q', isometric respectively to $\mathbf{P}^2(\mathbb{H}), \mathbf{P}^2(\mathbb{C}), \mathbf{P}^2(\mathbb{R}), \mathbf{S}^2$, containing both the center of the potential and an initial position of the particle such that an initial velocity of the particle is tangent to Q'. This means that the particle will always move along Q'. The standard transition from spaces $\mathbf{P}^2(\mathbb{C}a), \mathbf{P}^n(\mathbb{H}), \mathbf{P}^n(\mathbb{C}), \mathbf{S}^n$ to their noncompact analogous will give *mutatis mutandis* the integrability of a one-body motion on spaces $\mathbf{H}^2(\mathbb{C}a), \mathbf{H}^n(\mathbb{H}), \mathbf{H}^n(\mathbb{C}), \mathbf{H}^n(\mathbb{R})$ for all central potentials.

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6.1.1 The Motion on Spaces $\mathbf{P^2}(\mathbb{C}a), \mathbf{P^2}(\mathbb{H}), \mathbf{P^2}(\mathbb{C})$

Here we use notations from Sects. 1.2, 4.2.3, 4.3.2 and results of Chap. 3. Denote by Q one of the spaces $\mathbf{P}^2(\mathbb{C}a), \mathbf{P}^2(\mathbb{H}), \mathbf{P}^2(\mathbb{C})$ and by $M = T^*Q$ the phase space of the one-body problem on Q.

The symmetry group of the one-body problem in an arbitrary central potential is the subgroup of the isometry group of Q conserving a center $\mathbf{x}_0$ of a central potential V. This group was denoted above as K. The stationary subgroup of the group K, corresponding to a point $\mathbf{x} \in Q$, $\mathbf{x} \neq \mathbf{x}_0$, $\mathbf{x} \notin A_{\mathbf{x}_0}$, is the group K_0 conserving two points in Q in general position and the geodesic $\widetilde{\gamma}$, joining them (see Proposition 1.1). Let $\mathbf{x} = \widetilde{\gamma}(t)$, where t is the natural parameter. According to Lemma 5.2 one has

$$\bigcup_{t\in(0,\operatorname{diam} Q)} K\widetilde{\gamma}(t) = I \times (K/K_0) = Q',$$

where $I = (0, \operatorname{diam} Q)$, Q' is an open dense subspace in Q and

$$\operatorname{mes}(Q\backslash Q') = \operatorname{mes}\left((\mathbf{x}_0) \cup A_{\mathbf{x}_0}\right) = 0$$

for the measure mes on the space Q, induced by the Riemannian metric.

Let h_s be the Hamiltonian function for the one-body problem on Q, defined on $T^*Q' = T^*I \times T^*(K/K_0)$. It is K_0-invariant and therefore can be expressed through coordinates on T^*I and generators of the Poisson algebra $\mathcal{P}\left(T^*(K/K_0)\right)^K$ (cf. Sect. 4.3.2).

In notations of Proposition 1.2 the Lie algebra of the group K is $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$, where $\mathfrak{k}_0$ is the Lie algebra of the group K_0, $\mathfrak{k}_\lambda$ and $\mathfrak{k}_{2\lambda}$ are Ad_{K_0}-invariant subspaces in $\mathfrak{k}$, $\dim_{\mathbb{R}} \mathfrak{k}_\lambda = q_1$, $\dim_{\mathbb{R}} \mathfrak{k}_{2\lambda} = q_2$.

In order to find generators of the Poisson algebra $\mathcal{P}\left(T^*(K/K_0)\right)^K$, we are to calculate the base Ad_{K_0}-invariant elements in the commutative algebra $S(\mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda})$ (see Theorem 4.5). It was shown in Chap. 3 that restrictions of the Killing form $\mathrm{Kil}_{\mathfrak{g}}$ onto $\mathfrak{k}_\lambda$ and $\mathfrak{k}_{2\lambda}$ are the only invariants of $\mathrm{Ad}_{K_0}|_{\mathfrak{k}_\lambda}$ and $\mathrm{Ad}_{K_0}|_{\mathfrak{k}_{2\lambda}}$ actions respectively. Besides, the $\mathrm{Ad}_{K_0}|_{\mathfrak{k}_\lambda\oplus\mathfrak{k}_{2\lambda}}$ action is transitive on the direct product of two spheres $\mathbf{S}^{q_1-1} \times \mathbf{S}^{q_2-1}$ embedded in the standart way into $\mathfrak{k}_\lambda \oplus \mathfrak{k}_{2\lambda}$.

This implies that the Poisson algebra $\mathcal{P}\left(T^*(K/K_0)\right)^K$ is freely generated by elements p_2, p_5, which correspond to invariant operators D_2 and D_5 from (3.2). Since $[D_2, D_5] = 0$, the isomorphism (4.34) gives $[p_2, p_5]_P = 0$ and the Poisson algebra $\mathcal{P}\left(T^*(K/K_0)\right)^K$ is commutative. In other words, the space K/K_0 is commutative.

Due to Proposition 4.10 and formula (4.38) it is easy to derive the expression for the one-body Hamiltonian function h_s from the two-body quantum Hamiltonian in Sect. 5.4 in the limiting case of $m_2 = \infty, \beta = 0$. In this case $m = m_1$,

$$A_s = \frac{(1+r^2)^2}{4mR^2r^2},\ F_s = \frac{1+r^2}{mR^2r^2},\ B_s = C_s = D_s = E_s = 0\ ,$$

and one gets

$$h_s = \frac{(1+r^2)^2}{8mR^2} p_r^2 + \frac{1+r^2}{2mR^2r^2}\left(p_2 + \frac{1+r^2}{4} p_5\right) + V(r) \,, \qquad (6.1)$$

where $[p_2, p_5]_P = [p_2, r]_P = [p_2, r]_P = [p_5, r]_P = [p_5, p_r]_P = 0, [r, p_r]_P = 1$. Here and below we identify the functions p_2, p_5 on $T^*(K/K_0)$ with the functions $p_2 \circ \tilde{\pi}, p_5 \circ \tilde{\pi}$ on T^*Q' and the function V on Q' with $V \circ \pi_4$, where

$$\tilde{\pi} : T^*I \times T^*(K/K_0) \to T^*(K/K_0), \; \pi_4 : T^*Q' \to Q'$$

are canonical projections.

Assumption 4.1 is valid for $M = T^*Q'$, since Q' is a submanifold of an algebraic variety, K is an algebraic group and invariants p_2, p_5 are rational functions of algebraic coordinates on Q' and corresponding coordinates on fibers.

Thus, Remark 4.4 implies the noncommutative integrability of all Hamiltonian systems on $T^*(K/K_0)$ with Hamiltonian functions of the form $P(p_2, p_5)$, where P is a polynomial. Therefore, the one-particle system on Q with an arbitrary central potential is also integrable in noncommutative sense since the Hamiltonian function h_s itself is another independent integral of motion and $\dim T^*I = 2$.

The Hamiltonian reduction in this case is simple. The reduced phase space is T^*I and the reduced Hamiltonian function is (6.1) with $p_2, p_5 = \text{const}$. This reduced system describes the evolution of the distance between particle and the center of the potential.

In notations of Sects. 4.2.3 and 4.3.2 it holds $\mathcal{F}_2 = \mathcal{P}\left(T^*(K/K_0)\right)^K$ and commutativity of the Poisson algebra $\mathcal{F}_2$ gives $\operatorname{dind} \mathcal{F} = \operatorname{ddim} \mathcal{F}_2 = 2$ due to Remark 4.2. Therefore, a general common value set in $T^*(K/K_0)$ of independent functions from the Poisson algebra $\mathcal{F}$ is a 2-dimensional submanifold due to the same remark. Thus, a general common value set in T^*Q' of independent functions from $\mathcal{F}$ and the Hamiltonian function h_s is a 3-dimensional submanifold in T^*Q'. In this context the existence of nontrivial potentials, for which all bounded trajectories are closed, seems to be dubious. Below we shall see that for spaces $\mathbf{S}^2, \mathbf{P}^2(\mathbb{R})$ and $\mathbf{H}^2(\mathbb{R})$ such potentials exist.

6.1.2 One Particle Motion on $\mathbf{S}^2, \mathbf{P}^2(\mathbb{R})$ and $\mathbf{H}^2(\mathbb{R})$

Conserve here notations from the previous section and describe only the difference from it. In the present case we have $q_1 = 0, q_2 = 1, \mathfrak{k}_0 = \mathfrak{k}_\lambda = 0, \mathfrak{k} = \mathfrak{k}_{2\lambda}$; the group K is isomorphic to $\mathbf{S}^1$ and the group K_0 is trivial. The Poisson algebra $\mathcal{P}\left(T^*K\right)^K$ is generated by one element p_2, corresponding to the operator D_2 from Sect. 3.4 in the case $n = 2$.

Using the same limiting procedure as in the previous section one gets the one-body Hamiltonian function for spaces $\mathbf{S}^2, \mathbf{P}^2(\mathbb{R})$ in the form

$$h_s = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_2^2}{r^2}\right) + V(r) \,. \qquad (6.2)$$

Again the reduced space is T^*I and the reduced Hamiltonian function (6.2) with $p_2 = \text{const}$ describes the evolution of the distance between particle and the center of the potential.

Results for the hyperbolic plane are completely similar. Using the metric (1.25) one obtains the one-particle Hamiltonian function for the hyperbolic plane $\mathbf{H}^2(\mathbb{R})$ in the form

$$h_h = \frac{(1-r^2)^2}{8mR^2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r),\ r < 1\ . \tag{6.3}$$

6.2 Particle Motion in Bertrand Potentials on Constant Curvature Spaces

The content of this section was known more then a century ago, but *it is not well known*. The history and corresponding references can be found in Sect. 6.4.

Expression (6.2) can be easily obtained in more direct way. Indeed, due to (4.37) and (1.19) the Hamiltonian function for the natural one-body system in a central potential is obtained in the form (6.2), where p_2 is equal to the angular momentum p_φ, corresponding to the angular coordinate φ. Evidently, p_φ is an integral of motion.

Using metrics (1.20) and (1.26) rewrite Hamiltonian functions (6.2) and (6.3) in the forms

$$h_s = \frac{(1+v^2)^2}{2mR^2}p_v^2 + \frac{1+v^2}{2mv^2R^2}p_\varphi^2 + V(v),\ 0 < v < \infty \tag{6.4}$$

$$h_h = \frac{(1-v^2)^2}{2mR^2}p_v^2 + \frac{1-v^2}{2mv^2R^2}p_\varphi^2 + V(v),\ 0 < v < 1\ . \tag{6.5}$$

For Euclidean plane $\mathbf{E}^2$ with the metric $d\rho^2 + \rho^2 d\varphi^2$ one has the one-particle Hamiltonian function in the form

$$h_e = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2}\right) + V(\rho),\ 0 < \rho < \infty\ . \tag{6.6}$$

The *second Kepler law* for Euclidean plane is valid for all central potentials and means that *the line segment joining the point* $\rho = 0$ *and the particle position sweeps out equally large areas within equally long periods of time.* If one substitute words "*the line segment*" for "*the shortest geodesic segment*", this law will not be valid in spaces $\mathbf{S}^2, \mathbf{P}^2(\mathbb{R})$ and $\mathbf{H}^2(\mathbb{R})$. However, the following form of the second Kepler law is valid for spaces $\mathbf{E}^2$, $\mathbf{S}^2, \mathbf{P}^2(\mathbb{R})$ and $\mathbf{H}^2(\mathbb{R})$: *the geodesic segment starting from the center* O *of a potential in a direction of a particle position* P *and having the length two times more than the* $\mathrm{dist}(O,P)$ *sweeps out equally large areas within equally long periods of time.*[1]

Indeed, in the hyperbolic case the area of a thin geodesic sector, being swept out by the shortest geodesic segment of a length ρ, joining the point O with the point v, φ, is

[1] For the spaces $Q = \mathbf{S}^2, \mathbf{P}^2(\mathbb{R})$ one should additionally require that $\mathrm{dist}(O,P) < \frac{1}{2}\,\mathrm{diam}\,Q$ or consider the area, being swept out, with the corresponding multiplicity.

$$dS_v = R^2 d\varphi \int\limits_0^v \frac{vdv}{(1-v^2)^{3/2}} = R^2 \left(\frac{1}{\sqrt{1-v^2}} - 1 \right) d\varphi \ ,$$

where $d\varphi$ is an angle increment. Due to formula (1.27) if a coordinate $\tilde{v}$ corresponds to the similar geodesic segment of the length 2ρ, then $\tilde{v} = 2v/(1+v^2)$ and

$$dS_{\tilde{v}} = \frac{2R^2 v^2}{1-v^2} d\varphi$$

that implies

$$\frac{dS_{\tilde{v}}}{dt} = \frac{2R^2 v^2 \dot{\varphi}}{1-v^2} = \frac{2p_\varphi}{m} = \text{const} \, .$$

The proof in the spherical case is completely similar and in Euclidean case is evident.

It is well known [5, 8] that there are two central special potentials in Euclidean space $\mathbf{E}^n$, the Coulomb and the isotropic harmonic oscillator ones, for which all finite trajectories of a classical particle are closed, provided that they exist. The problem of finding such potentials is known as *Bertrand* one and was solved by Bertrand in [15]. We will refer such potentials as *Bertrand* ones.

Consider the generalization of the Bertrand problem onto constant curvature spaces. Here we will reduce this generalization to the original Bertrand problem in Euclidean space.

Let $w = -1/(vR)$ for the spaces $\mathbf{S}^2$, $\mathbf{H}^2(\mathbb{R})$ and $w = -1/\rho$ for Euclidean plane $\mathbf{E}^2$. Straightforward calculations for Hamiltonian functions (6.4), (6.5) and (6.6) lead to the equation

$$p_\varphi \frac{dw}{d\varphi} = \sqrt{2m(E - V(w)) - p_\varphi^2 (w^2 + \kappa)}, \; p_\varphi = \text{const},$$

where κ is the curvature: $\kappa = 0$ for $\mathbf{E}^2$, $\kappa = 1/R^2$ for $\mathbf{S}^2$ and $\kappa = -1/R^2$ for $\mathbf{H}^2(\mathbb{R})$; E is a constant value of the Hamiltonian function. Thus, the equation for trajectories in coordinates w, φ is

$$\varphi = \int \frac{p_\varphi dw}{\sqrt{2m(E - V(w)) - p_\varphi^2 (w^2 + \kappa)}} \, . \tag{6.7}$$

Since the constant term $p_\varphi^2 \kappa$ can be united with the arbitrary constant term $2E$, the form of Bertrand potentials in all three cases are the same w.r.t. the variable w.

For Euclidean space the solutions of the Bertrand problem is $V_c = \gamma w$ (Coulomb potential) and $V_o = \gamma w^{-2}$, $\gamma > 0$ (isotropic harmonic oscillator potential) [5, 8]. Thus, one gets generalizations of these potentials: $V_c = -\gamma/(vR)$ (Coulomb potential) and $V_o = \gamma v^2 R^2$ (isotropic oscillator potential) for spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$.

Remark 6.1. *Note that some authors* [12, 73] *normalize the potential* V_c *for the space* $\mathbf{H}^n(\mathbb{R})$ *in such a way that* $V_c \to 0$ *on the absolute, i.e., as* $v \to 1$*. This gives the expression*

$$V_c = \frac{\gamma}{R}\left(1 - \frac{1}{v}\right).$$

In this case the summand γ/R should be included in all corresponding formulas below as the energy shift.

The singularity of the oscillator potential divides the sphere $\mathbf{S}^2$ into two separate domains of motion along the equator $\rho = \pi R/2$. For the factor space $\mathbf{P}^2(\mathbb{R}) \cong \mathbf{S}^2/\,\mathrm{Z}_2$ this singularity is concentrated in one point, most far from the potential center.

6.2.1 The Kepler Problem

Consider first the motion in the potential V_c or the *Kepler problem.* Let ρ be the distance from the point $v = 0$. Using formulas (1.21) and (1.27) one can normalize V_c in the following way

$$V_{c,h} = -\frac{\gamma m}{R}\coth\frac{\rho}{R},\; V_{c,s} = -\frac{\gamma m}{R}\cot\frac{\rho}{R},\; \gamma > 0\;,$$

where subscript "h" means "hyperbolic", subscript "s" means "spherical" and it holds

$$V_{c,h},\, V_{c,s} \to V_{c,e} := -\gamma m/\rho \quad \text{as} \quad R \to +\infty\;.$$

As in Euclidean case the Coulomb potential $V_c(\rho)$ in spaces $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ is inversely proportional to the area of a sphere of the radius ρ.

Renormalizing if necessarily the energy E and the momentum p_φ we will suppose below $m = 1$. Calculating the integral in (6.7) one gets the particle trajectory in the form

$$v(\varphi) = \frac{p}{1 + e\cos\varphi},\; p := \frac{p_\varphi^2}{\gamma R},\; e := \sqrt{1 + \frac{2p_\varphi^2}{\gamma^2}\left(E - \frac{\kappa p_\varphi^2}{2}\right)}, \tag{6.8}$$

where the null value of φ corresponds to the pericentre. This form of the trajectory coincides with the polar equation for conics (ellipse, hyperbola and parabola) on Euclidean plane. Below we shall see that this similarity is more deep.

Kepler Problem in the Hyperbolic Space

For the particles trajectory one has

$$2\left(E + \frac{\gamma}{vR}\right) - \frac{p_\varphi^2}{R^2}\left(\frac{1}{v^2} - 1\right) \geqslant 0,\ \text{for } 0 \leqslant v < 1$$

and therefore

$$E \geqslant \widetilde{V}(v) := \frac{p_\varphi^2}{2R^2}\left(\frac{1}{v^2} - 1\right) - \frac{\gamma}{vR} \geqslant \widetilde{V}_{\min} = \begin{cases} -\dfrac{\gamma}{R},\ \text{if } \dfrac{p_\varphi^2}{\gamma R} \geqslant 1, \\ -\dfrac{p_\varphi^2}{2R^2} - \dfrac{\gamma^2}{2p_\varphi^2},\ \text{if } \dfrac{p_\varphi^2}{\gamma R} < 1\,. \end{cases}$$

Note that $\dfrac{p}{1+e} < 1$ for every admissible values of E, p_φ and the trajectory defined by (6.8) (in the unit circle $v < 1$) is connected. In the case $p_\varphi^2/(\gamma R) \geqslant 1$ the function $\widetilde{V}(v)$ has no local minima for $0 < v < 1$ and the motion is infinite for all $E \geqslant -\gamma/R$. In the case $p_\varphi^2/(\gamma R) < 1$ the motion is finite (i.e., the trajectory is compact in the unit circle $v < 1$) if

$$-p_\varphi^2/(2R^2) - \gamma^2/(2p_\varphi^2) \leqslant E < -\gamma/R \tag{6.9}$$

and infinite if $E \geqslant -\gamma/R$.

An ellipse on Euclidean plane $\mathbf{E}^2$ is defined as the set of points $\mathbf{x} \in \mathbf{E}^2$ for those

$$\mathrm{dist}(\mathbf{x}, \mathbf{f}_1) + \mathrm{dist}(\mathbf{x}, \mathbf{f}_2) = 2a = \mathrm{const} \tag{6.10}$$

for two focuses $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{E}^2$ such that $2c := \mathrm{dist}(\mathbf{f}_1, \mathbf{f}_2) < 2a$, see Fig. 6.1.

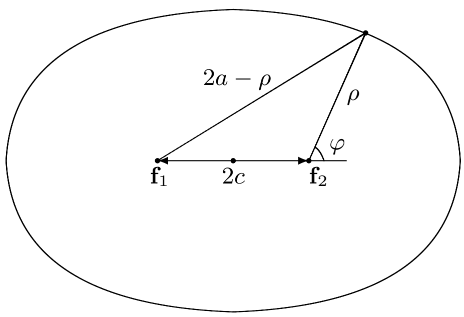

Fig. 6.1. Ellipse

It is easily verified using the cosine theorem that

$$\rho(\varphi) = \frac{p_e}{1 + \varepsilon_e \cos\varphi}, \; p_e = \frac{a^2 - c^2}{a}, \; \varepsilon_e = \frac{c}{a} \; .$$

Consider the same definition of an *ellipse on the hyperbolic plane* $\mathbf{H}^2(\mathbb{R})$ (see again Fig. 6.1). The hyperbolic cosine theorem now gives

$$\cosh\frac{2a-\rho}{R} = \cosh\frac{2c}{R}\cosh\frac{\rho}{R} + \sinh\frac{2c}{R}\sinh\frac{\rho}{R}\cos\varphi$$

that implies

$$v(\varphi) = \tanh\frac{\rho}{R} = \frac{p_h}{1 + \varepsilon_h \cos\varphi} \; , \tag{6.11}$$

for

$$p_h = \frac{\cosh\dfrac{2a}{R} - \cosh\dfrac{2c}{R}}{\sinh\dfrac{2a}{R}} \; , \; \varepsilon_h = \frac{\sinh\dfrac{2c}{R}}{\sinh\dfrac{2a}{R}} \; . \tag{6.12}$$

Due to its definition an ellipse is a finite curve and therefore $\dfrac{p_h}{1-\varepsilon_h} < 1$ or

$$p_h + \varepsilon_h < 1 \; . \tag{6.13}$$

On the other hand, if (6.13) is fulfilled, then (6.11) defines an ellipse with some $a > c > 0$ derived from (6.12). Note that for parameters p and e defined in (6.8) the inequality $p + e < 1$ is equivalent to $E < -\gamma/R$.

Thus, one sees that the particle trajectory defined by (6.8) for $E < -\gamma/R$ is an ellipse with one focus in the centre of the potential V_c. This is the *first Kepler law* for the hyperbolic plane.

It is easily seen that for this ellipse it holds

$$R \tanh \frac{2a}{R} = R \frac{v_{\max} + v_{\min}}{1 + v_{\max} v_{\min}} = -\frac{\gamma}{E} \ , \tag{6.14}$$

where $v_{\max} = v|_{\varphi=\pi} = p/(1-e)$, $v_{\min} = v|_{\varphi=0} = p/(1+e)$ and the major ellipse semiaxis a does not depend upon the momentum p_φ.

The period T of the particle motion along the elliptic orbit can be obtained as

$$\begin{aligned} T &= 2 \int\limits_{v_{\min}}^{v_{\max}} \frac{R^2 dv}{(1-v^2)\sqrt{2R^2(E + \dfrac{\gamma}{Rv}) + p_\varphi^2 \dfrac{v^2-1}{v^2}}} \\ &= \frac{\pi R}{\sqrt{2}} \left(\frac{1}{\sqrt{-\dfrac{\gamma}{R} - E}} - \frac{1}{\sqrt{\dfrac{\gamma}{R} - E}} \right) . \end{aligned}$$

Its independence on p_φ is of a general nature. Indeed, due to the Gordon theorem [51] (see also [58], Chap. IV) if all trajectories of a Hamiltonian system lying on some energy level $E = \text{const}$ are closed, then their periods are the same. Using (6.14) one can express the period T through the major semiaxis a:

$$T = \frac{\pi R^{3/2}}{\sqrt{2\gamma}} \left(\frac{1}{\sqrt{\coth \dfrac{2a}{R} - 1}} - \frac{1}{\sqrt{\coth \dfrac{2a}{R} + 1}} \right) = \frac{2\pi}{\sqrt{\gamma}} \sqrt{R^3 \sinh^3 \frac{a}{R} \cosh \frac{a}{R}} \ . \tag{6.15}$$

This formula is known as the *third Kepler law* for the hyperbolic plane [103, 105].

Consider now infinite trajectories. The Beltrami-Klein model (see Sect. 1.3.3) of the hyperbolic plane is based upon the one-to-one correspondence between points of a one sheet of the two-sheet hyperboloid (1.22) in space $\mathbb{R}^3$ endowed with the Minkowski metric (1.23) (for $n = 3$) and points of the circle

$$D_R := \left(\mathbf{x} \in \mathbb{R}^3 \middle| \ x_3 = R, \, x_1^2 + x_2^2 < R^2 \right) \ .$$

This correspondence is established by the central projection from the point $0 \in \mathbb{R}^3$. This means that Euclidean conics in D_R (in particular defined by the equation $v = p_h/(1 + \varepsilon_h \cos\varphi)$, for $x_1 = Rv\cos\varphi, x_2 = Rv\sin\varphi$) correspond to the curves on this hyperboloid, which are cut by cones in $\mathbb{R}^3$ with their common vertex at the point $0 \in \mathbb{R}^3$. Such curves on the hyperboloid and their images in D_R can be considered as *conics on the hyperbolic plane.* The more detailed description of conics on the hyperbolic plane can be found in [77, 90, 105, 149] and [186].[2]

By full analogy with Euclidean case a *hyperbola in* $\mathbf{H}^2(\mathbb{R})$ is a set of points $\mathbf{x}$ for those

$$|\mathrm{dist}(\mathbf{x}, \mathbf{f}_1) - \mathrm{dist}(\mathbf{x}, \mathbf{f}_2)| = 2a = \mathrm{const} \tag{6.16}$$

for two focuses $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{H}^2(\mathbb{R})$ such that $2c := \mathrm{dist}(\mathbf{f}_1, \mathbf{f}_2) > 2a$, see Fig. 6.2.

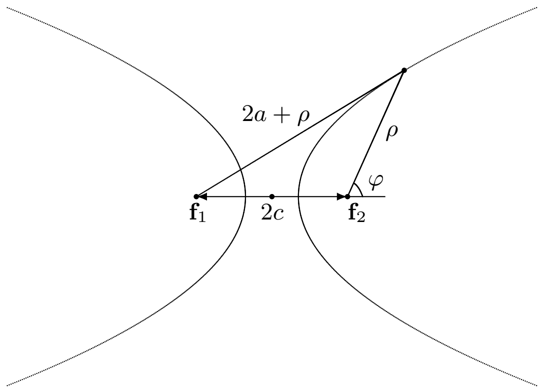

Fig. 6.2. Hyperbola

Using the cosine theorem again one gets (cf. (6.11)) for the right focus $\mathbf{f}_2$ in the point $v = 0$ the equation

$$v = \tanh \frac{\rho}{R} = \frac{p_h}{1 - \varepsilon_h \cos\varphi}$$

for the right hyperbola branch and the equation

$$v = -\frac{p_h}{1 + \varepsilon_h \cos\varphi}$$

for the left one, where

$$p_h = \frac{\cosh \frac{2c}{R} - \cosh \frac{2a}{R}}{\sinh \frac{2a}{R}}, \ \varepsilon_h = \frac{\sinh \frac{2c}{R}}{\sinh \frac{2a}{R}} > 1 \, . \tag{6.17}$$

The hyperbola definition implies $-p_h/(1 - \varepsilon_h) < 1$, i.e.,

$$\varepsilon_h - p_h > 1 \, . \tag{6.18}$$

Conversely, parameters p_h and ε_h obeying (6.18), correspond to some hyperbola with $c > a > 0$, defined from (6.17).

[2] It seems to be probable that the classification of conics on the hyperbolic plane was found by W. Story in 1882 [186]. In any case the claim in [77] that it was found by H. Liebmann in [105] is evidently erroneous.

If both inequalities (6.13) and (6.18) are violated, then (6.11) defines a one branch infinite conic with the focus at $v = 0$. In the boundary case

$$\varepsilon_h + p_h = 1 \tag{6.19}$$

this curve is called *horoellipse* [42] or *elliptic parabola* [105] since it is a "big ellipse" with one point on the absolute (see Fig. 6.3), coinciding with the second focus.

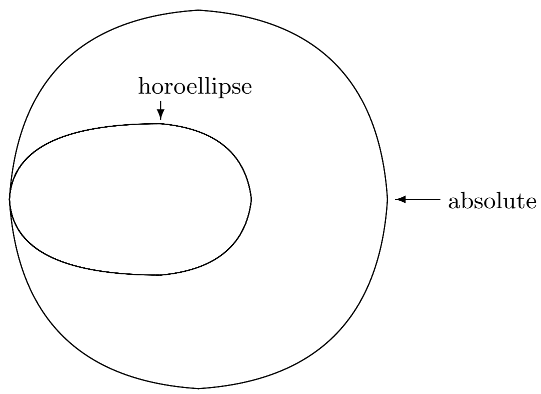

Fig. 6.3. Horoellipse

In another boundary case

$$\varepsilon_h - p_h = 1 \tag{6.20}$$

one obtains *horohyperbola* [42] or *hyperbolic parabola* [105] with the "second focus" on the absolute. Other one-branch infinite curves (6.11) corresponding to the inequality

$$1 - p_h < \varepsilon_h < 1 + p_h \tag{6.21}$$

are called in [42, 105, 186] *semihyperbolas*.

Using formulas (6.18)–(6.21), by direct computations one can easily verify that the particle trajectory is a horoellipse if $E = -\gamma/R$, a semihyperbola if $-\gamma/R < E < \gamma/R$, a horohyperbola if $E = \gamma/R$ and a hyperbola if $E > \gamma/R$.

The additional integral for the Kepler problem, responsible for the closure of all bounded trajectories, can be found as a polynomial w.r.t momenta p_v and p_φ of the second degree in one of the two equivalent forms:

$$I_{1,h} = \left(\frac{p_\varphi^2}{Rv} - \gamma\right)\cos\varphi + \frac{(1-v^2)p_v p_\varphi}{R}\sin\varphi \, ,$$
$$I_{2,h} = \left(\frac{p_\varphi^2}{Rv} - \gamma\right)\sin\varphi - \frac{(1-v^2)p_v p_\varphi}{R}\cos\varphi \, .$$

Note that $\partial I_{2,h}/\partial\varphi = I_{1,h}$. These expressions are analogs for the hyperbolic plane of components of the *Runge-Lenz vector* in Euclidean case. It is easily verified that integrals $p_\varphi, I_{1,h}, I_{2,h}$ and the Hamiltonian function itself

$$h = \frac{(1-v^2)^2}{2R^2} p_v^2 + \frac{1-v^2}{2v^2R^2} p_\varphi^2 - \frac{\gamma}{R} \coth \frac{\rho}{R}$$

are connected by the identity

$$I_{1,h}^2 + I_{2,h}^2 = \gamma^2 + 2p_\varphi^2 \left(h + \frac{p_\varphi^2}{2R^2} \right) .$$

Kepler Problem on the Sphere

In the spherical case the coordinate ρ is more convenient than v, since the latter is not continuous on the equator of the sphere. The Hamiltonian function for $m = 1$ now is

$$h = \frac{1}{2} \left(p_\rho^2 + \frac{p_\varphi^2}{R^2 \sin^2 \rho/R} \right) - \frac{\gamma}{R} \cot \frac{\rho}{R} .$$

An ellipse on the sphere $\mathbf{S}^2$ is defined by (6.10) as on Euclidean and the hyperbolic planes above. Now one can suppose that $2a \leqslant \pi R$. Otherwise, if $\pi R < 2a < 2\pi R$, consider diametrically opposite (w.r.t. focuses $\mathbf{f}_1$ and $\mathbf{f}_2$) points $\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2$ as a new pair of focuses. Then $\operatorname{dist}(\mathbf{x}, \mathbf{f}_i) = \pi R - \operatorname{dist}(\mathbf{x}, \tilde{\mathbf{f}}_i)$, $i = 1, 2$ and

$$\operatorname{dist}(\mathbf{x}, \tilde{\mathbf{f}}_1) + \operatorname{dist}(\mathbf{x}, \tilde{\mathbf{f}}_2) = 2\pi R - 2a < \pi R .$$

In the case $2a = 2c = \pi R$ any point of the sphere satisfies (6.10). In the case $2c < 2a = \pi R$ this equation defines the big circle passing through focuses. In the general case $0 \leqslant 2c < 2a < \pi R$ from the spherical cosine theorem one has (see Fig. 6.1):

$$\cos \frac{2a - \rho}{R} = \cos \frac{2c}{R} \cos \frac{\rho}{R} - \sin \frac{2c}{R} \sin \frac{\rho}{R} \cos \varphi$$

that implies

$$\tan \frac{\rho}{R} = \frac{\cos \frac{2c}{R} - \cos \frac{2a}{R}}{\sin \frac{2a}{R} + \sin \frac{2c}{R} \cos \varphi} = \frac{p_s}{1 + \varepsilon_s \cos \varphi} , \tag{6.22}$$

where

$$p_s = \frac{\cos \frac{2c}{R} - \cos \frac{2a}{R}}{\sin \frac{2a}{R}} > 0, \ \varepsilon_s = \frac{\sin \frac{2c}{R}}{\sin \frac{2a}{R}} > 0 . \tag{6.23}$$

If one replaces one of the ellipse focus by the diametrically opposite point, he obtains the hyperbola (6.16) and vise versa. This means that in the spherical case ellipses and hyperbolas coincide.

For the particle trajectory one gets the inequality

$$E \geqslant \widetilde{V}(\rho) := \frac{p_\varphi^2}{2R^2} \left(\cot^2 \frac{\rho}{R} + 1 \right) - \frac{\gamma}{R} \cot \frac{\rho}{R}, \ 0 < \rho < \pi R .$$

The function $\widetilde{V}(\rho)$ has one minimum

$$\widetilde{V}_{\min} = \frac{p_\varphi^2}{2R^2} - \frac{\gamma^2}{2p_\varphi^2}$$

and $\widetilde{V}(\rho) \to +\infty$ as $\rho \to 0$ or $\rho \to \pi R$. Thus, for $E \geqslant \widetilde{V}_{\min}$ this yields the elliptic trajectory (if $p_\varphi \neq 0$), defined by the equation

$$\tan\frac{\rho}{R} = \frac{p}{1 + e\cos\varphi},\ p = \frac{p_\varphi^2}{\gamma R},\ e = \sqrt{1 + \frac{2p_\varphi^2}{\gamma^2}\left(E - \frac{p_\varphi^2}{2R^2}\right)},$$

which is the *first Kepler law* for the sphere $\mathbf{S}^2$. In the case $E = \widetilde{V}_{\min}$ the trajectory is a circle defined by $\tan\frac{\rho}{R} = p$.

For the major ellipse axis $2a$ one has

$$\tan\frac{2a}{R} = \tan\left(\frac{\rho_{\max} + \rho_{\min}}{R}\right) = \frac{\tan\frac{\rho_{\max}}{R} + \tan\frac{\rho_{\min}}{R}}{1 - \tan\frac{\rho_{\max}}{R}\tan\frac{\rho_{\min}}{R}} = -\frac{\gamma}{RE}\ .$$

This means that for negative energies the major ellipse axis is shorter than a quarter of a big circle ($2a < \pi R/2$) and for positive energies one has $\pi R/2 < 2a < \pi R$. Similarly to (6.15) for the period one gets the equality

$$T = \frac{2\pi}{\sqrt{\gamma}}\sqrt{R^3 \sin^3\frac{a}{R}\cos\frac{a}{R}}\ ,$$

which is known as the *third Kepler law* for the sphere $\mathbf{S}^2$ (W. Killing, 1885, [87]).

Now the additional integrals are

$$I_{1,s} = \left(\frac{p_\varphi^2}{R}\cot\frac{\rho}{R} - \gamma\right)\cos\varphi + p_\rho p_\varphi \sin\varphi\ ,$$
$$I_{2,s} = \left(\frac{p_\varphi^2}{R}\cot\frac{\rho}{R} - \gamma\right)\sin\varphi - p_\rho p_\varphi \cos\varphi\ .$$

These expressions are spherical analogs of components of the Runge-Lenz vector in Euclidean case. Again it holds $\partial I_{2,h}/\partial\varphi = I_{1,h}$. The identity, connecting integrals $p_\varphi, I_{1,s}, I_{2,s}$ and h is:

$$I_{1,s}^2 + I_{2,s}^2 = \gamma^2 + 2p_\varphi^2\left(h - \frac{p_\varphi^2}{2R^2}\right)\ .$$

6.2.2 The Isotropic Oscillator Problem

Normalize the isotropic oscillator potential in such a way that it converges to the Euclidean expression $\omega^2\rho^2/2$ as $R \to +\infty$

$$V_{o,h} = \frac{1}{2}\omega^2 R^2 v^2 = \frac{1}{2}\omega^2 R^2 \tanh^2\frac{\rho}{R},\ V_{o,s} = \frac{1}{2}\omega^2 R^2 v^2 = \frac{1}{2}\omega^2 R^2 \tan^2\frac{\rho}{R}\ .$$

Calculating the integral in (6.7) one gets the particle trajectory in the form

$$v^2(\varphi) = \frac{p_\varphi^2/R^2}{E - \frac{1}{2}p_\varphi^2\kappa - \sqrt{(E - \frac{1}{2}p_\varphi^2\kappa)^2 - \omega^2 p_\varphi^2 \cos 2\varphi}} \, . \tag{6.24}$$

We shall show below that this equation defines an ellipse with its center at the point $v = 0$ or an unbounded curve (the latter is possible only in the hyperbolic case).

The Isotropic Oscillator Problem on the Hyperbolic Plane

For the particles trajectory one has the inequality

$$E \geqslant \widetilde{V}(v) := \frac{p_\varphi^2}{2R^2}\left(\frac{1}{v^2} - 1\right) + \frac{1}{2}\omega^2 v^2 R^2 \geqslant \widetilde{V}_{\min} = \begin{cases} \omega p_\varphi - \dfrac{p_\varphi^2}{2R^2}, \text{ if } p_\varphi < \omega R^2, \\ \dfrac{\omega^2 R^2}{2}, \text{ if } p_\varphi \geqslant \omega R^2 \, . \end{cases}$$

The motion is finite iff

$$p_\varphi < \omega R^2, \, \omega p_\varphi - \frac{p_\varphi^2}{2R^2} \leqslant E < \frac{\omega^2 R^2}{2} \tag{6.25}$$

and infinite for $E \geqslant \omega^2 R^2/2$. The period for the finite motion is

$$T = \frac{2\pi R}{\sqrt{\omega^2 R^2 - 2E}} \, .$$

The curve defined by (6.24) for $\kappa = -R^{-2}$ is an ellipse with the point $v = 0$ in its center, provided the curve is finite. Indeed, consider Fig. 6.4,

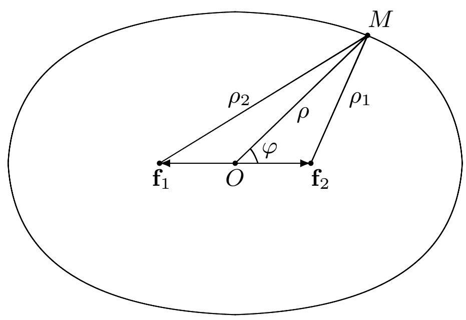

Fig. 6.4. Ellipse

where $\rho_1 := \mathrm{dist}(\mathbf{f}_1, M), \rho_2 := \mathrm{dist}(\mathbf{f}_2, M), \rho_1 + \rho_2 = 2a = \mathrm{const}, \rho := \mathrm{dist}(O, M), \mathrm{dist}(O, \mathbf{f}_1) = \mathrm{dist}(O, \mathbf{f}_2) = c$.

The cosine theorem implies

$$\begin{aligned} \cosh\frac{\rho_1}{R} &= \cosh\frac{\rho}{R}\cosh\frac{c}{R} - \sinh\frac{\rho}{R}\sinh\frac{c}{R}\cos\varphi \, , \\ \cosh\frac{\rho_2}{R} &= \cosh\frac{\rho}{R}\cosh\frac{c}{R} + \sinh\frac{\rho}{R}\sinh\frac{c}{R}\cos\varphi \, . \end{aligned} \tag{6.26}$$

Besides, it holds the identity

$$\begin{aligned}\Big(\cosh\frac{\rho_1+\rho_2}{R} &- \cosh\frac{\rho_1}{R}\cosh\frac{\rho_2}{R}\Big)^2 \qquad (6.27)\\ &= \sinh^2\frac{\rho_1}{R}\sinh^2\frac{\rho_2}{R}\\ &= \Big(1-\cosh^2\frac{\rho_1}{R}\Big)\Big(1-\cosh^2\frac{\rho_2}{R}\Big)\\ &= \Big(1+\cosh\frac{\rho_1}{R}\cosh\frac{\rho_2}{R}\Big)^2 - \Big(\cosh\frac{\rho_1}{R}+\cosh\frac{\rho_2}{R}\Big)^2 .\end{aligned}$$

Substituting expressions (6.26) for $\cosh\rho_1/R, \cosh\rho_2/R$ and the expression $\rho_1+\rho_2 = 2a$ into (6.27), one gets

$$\cosh^2\frac{a}{R}\sinh^2\frac{a}{R} = \cosh^2\frac{c}{R}\sinh^2\frac{a}{R}\cosh^2\frac{\rho}{R} - \sinh^2\frac{c}{R}\cosh^2\frac{a}{R}\sinh^2\frac{\rho}{R}\cos^2\varphi . \tag{6.28}$$

If $\varphi = \pi/2$, then $\rho_1 = \rho_2 = a, \rho = b$, where b is the small semiaxis of this ellipse and $\cosh a/R = \cosh c/R \cosh b/R$. Excluding c from (6.28) with the help of the latter equation, one can obtain the ellipse equation in the form:

$$\frac{\tanh^2(\rho/R)}{\tanh^2(a/R)}\cos^2\varphi + \frac{\tanh^2(\rho/R)}{\tanh^2(b/R)}\sin^2\varphi = 1 , \tag{6.29}$$

which is the analogue of the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{\rho^2}{a^2}\cos^2\varphi + \frac{\rho^2}{b^2}\sin^2\varphi = 1$$

for an ellipse on Euclidean plane. Another version of (6.29) is

$$\tanh^2\frac{\rho}{R} = \frac{2\tanh^2(a/R)\tanh^2(b/R)}{\tanh^2(a/R)+\tanh^2(b/R) - \big(\tanh^2(a/R)-\tanh^2(b/R)\big)\cos 2\varphi}$$

that coincides with (6.24) for $\kappa = -R^{-2}$, provided (6.25) is fulfilled.

Note that

$$\tanh\frac{a}{R} = \tanh\frac{\rho}{R}\Big|_{\varphi=0} , \ \tanh\frac{b}{R} = \tanh\frac{\rho}{R}\Big|_{\varphi=\frac{\pi}{2}} .$$

Thus, from (6.24) one finds

$$\begin{aligned}\tanh^2\frac{a}{R} &= v^2(0) = \frac{p_\varphi^2/R^2}{E+\frac{1}{2}p_\varphi^2R^{-2} - \sqrt{(E+\frac{1}{2}p_\varphi^2R^{-2})^2-\omega^2p_\varphi^2}} ,\\ \tanh^2\frac{b}{R} &= v^2(\frac{\pi}{2}) = \frac{p_\varphi^2/R^2}{E+\frac{1}{2}p_\varphi^2R^{-2} + \sqrt{(E+\frac{1}{2}p_\varphi^2R^{-2})^2-\omega^2p_\varphi^2}} .\end{aligned}$$

This leads to the equation

$$\tanh\frac{a}{R}\tanh\frac{b}{R} = \frac{p_\varphi}{\omega R^2},$$

which means that the function of semiaxes on the left hand side does not depend upon the energy E.

Infinite trajectories of the isotropic oscillator on $\mathbf{H}^2(\mathbb{R})$ consist of two symmetric components and have no analogs in Euclidean limit.

The additional integral for the isotropic oscillator problem, responsible for the closure of all bounded trajectories, can be found as above for the Kepler problem in one of the two equivalent forms:

$$J_{1,h} = \left(\frac{p_\varphi^2}{R^2 v^2} - \frac{p_v^2 (1-v^2)^2}{R^2} - \omega^2 R^2 v^2 \right) \cos 2\varphi + 2 \frac{p_v p_\varphi}{R^2 v} (1-v^2) \sin 2\varphi \,,$$

$$J_{2,h} = \left(\frac{p_\varphi^2}{R^2 v^2} - \frac{p_v^2 (1-v^2)^2}{R^2} - \omega^2 R^2 v^2 \right) \sin 2\varphi - 2 \frac{p_v p_\varphi}{R^2 v} (1-v^2) \cos 2\varphi \,.$$

These expressions are analogous to components of the *Fradkin tensor* in Euclidean case. Integrals $p_\varphi, J_{1,h}, J_{2,h}$ and the Hamiltonian function itself

$$h = \frac{(1-v^2)^2}{2R^2} p_v^2 + \frac{1-v^2}{2v^2 R^2} p_\varphi^2 + \frac{1}{2} \omega^2 v^2 R^2$$

are connected by the identity

$$J_{1,h}^2 + J_{2,h}^2 = \left(2h + \frac{p_\varphi^2}{R^2} \right)^2 - 4 p_\varphi^2 \omega^2 \,.$$

The Isotropic Oscillator Problem in the Spherical Case

The Hamiltonian function for $m = 1$ now is

$$h = \frac{1}{2} \left(p_\rho^2 + \frac{p_\varphi^2}{R^2 \sin^2 \rho / R} \right) + \frac{1}{2} \omega^2 R^2 \tan^2 \frac{\rho}{R} \,.$$

The inequality for the energy has the form

$$E \geqslant \widetilde{V}(\rho) := \frac{p_\varphi^2}{2R^2} \left(\cot^2 \frac{\rho}{R} + 1 \right) + \frac{1}{2} \omega^2 R^2 \tan^2 \frac{\rho}{R}$$

and thus

$$E \geqslant \widetilde{V}_{\min} = p_\varphi \omega + \frac{p_\varphi^2}{2R^2} \,.$$

For all these energies the motion occurs on a hemisphere $\rho \leqslant \pi R/2$, since $\tan^2 \dfrac{\rho}{R} \to +\infty$ as $\rho \to \pi R/2$. Due to this fact one can consider this motion also as the motion on the projective plane $\mathbf{P}^2(\mathbb{R})$. Its period is

$$T = \frac{2\pi R}{\sqrt{\omega^2 R^2 + 2E}} \,.$$

Now all trajectories are ellipses with their centers at the point $\rho = 0$. By full analogy with the hyperbolic case one gets for such ellipses the following equivalent equations:

$$\frac{\tan^2(\rho/R)}{\tan^2(a/R)}\cos^2\varphi+\frac{\tan^2(\rho/R)}{\tan^2(b/R)}\sin^2\varphi=1\ ,$$

$$\tan^2\frac{\rho}{R}=\frac{2\tan^2(a/R)\tan^2(b/R)}{\tan^2(a/R)+\tan^2(b/R)-\bigl(\tan^2(a/R)-\tan^2(b/R)\bigr)\cos 2\varphi}\ ,$$

where a and b are the ellipse semiaxes. Similarly to the hyperbolic case it holds

$$\tan\frac{a}{R}\tan\frac{b}{R}=\frac{p_\varphi}{\omega R^2}.$$

Now the additional integrals (spherical analogs of the Fradkin tensor) are:

$$J_{1,s}=\left(\frac{p_\varphi^2}{R^2}\cot^2\frac{\rho}{R}-p_\rho^2-\omega^2R^2\tan^2\frac{\rho}{R}\right)\cos 2\varphi+2\frac{p_\rho p_\varphi}{R}\cot\frac{\rho}{R}\sin 2\varphi\ ,$$

$$J_{2,s}=\left(\frac{p_\varphi^2}{R^2}\cot^2\frac{\rho}{R}-p_\rho^2-\omega^2R^2\tan^2\frac{\rho}{R}\right)\sin 2\varphi-2\frac{p_\rho p_\varphi}{R}\cot\frac{\rho}{R}\cos 2\varphi\ .$$

The identity, connecting integrals $p_\varphi, J_{1,s}, J_{2,s}$ and the Hamiltonian function h has the form

$$J_{1,h}^2+J_{2,h}^2=\left(2h-\frac{p_\varphi^2}{R^2}\right)^2-4p_\varphi^2\omega^2\ .$$

6.3 Quantum Mechanical One-Body Problem for Bertrand Potentials on Constant Curvature Spaces

Similar to Euclidean case the quantum mechanical one-body problem for Bertrand potentials on spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$ inherits many symmetric properties from the classical one.

Note first off all that for both spaces and $n=3$ it holds

$$\triangle(V_c)=4\pi\gamma\delta\ , \tag{6.30}$$

where δ is a distribution, well-known as the delta-function, centered at the center of the potential V_c. This formula can be derived by the same calculations as in Euclidean case. Equation (6.30) is another motivation of introducing the potential V_c, independent on the generalized Bertrand problem. For the oscillator potential in Euclidean case it holds $\triangle(V_o)=\text{const}$, but it is not valid for spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$.

Consider the quantum mechanical spectral problem

$$H_V\psi=-\frac{1}{2}\triangle\psi+V\psi=E\psi\ ,$$

for Bertrand potentials V.

6.3.1 The Hyperbolic Case

Theorem 2.11 implies the self-adjointness of the operator H_{V_c} for $n \geqslant 2$ and $V_1 = 0$ with its domain $\mathrm{Dom}(H_V)$ given by (2.28). The same theorem implies the self-adjointness of H_{V_o} for $n \geqslant 2$ and $V_2 = 0$ with domain (2.28). Note that in all cases, except H_{V_c} for $n = 2$, Theorem 2.10 also can be applied. Formulae (1.26) and (2.16) yield the Laplace-Beltrami operator in the form

$$\triangle = \frac{(1-v^2)^2}{R^2}\left(\frac{\partial^2}{\partial v^2} + \left(\frac{n-1}{v} + \frac{(n-3)v}{1-v^2}\right)\frac{\partial}{\partial v}\right) + \frac{1-v^2}{R^2 v^2}\triangle_s,\ 0 < v < 1\ , \tag{6.31}$$

where $\triangle_s$ is the Laplace-Beltrami operator on the sphere $\mathbf{S}^{n-1}$ with the standard metric $\tilde{g}_s$.

Similarly to (2.38) one has the following isomorphism of the Hilbert spaces

$$\mathcal{L}^2(\mathbf{H}^n(\mathbb{R}), d\mu) \cong \mathcal{L}^2((0,1), d\mu_1) \otimes \mathcal{L}^2\left(\mathbf{S}^{n-1}, d\mu_{\mathbf{S}^{n-1}}\right),$$

where μ is the measure on $\mathbf{H}^n(\mathbb{R})$ defined by (2.25) for the metric (1.26),

$$d\mu_1 = \frac{Rv^{n-1}}{(1-v^2)^{\frac{n+1}{2}}}dv$$

and $d\mu_{\mathbf{S}^{n-1}}$ is the measure on $\mathbf{S}^{n-1}$, defined by (2.25) for the metric $\tilde{g}_s$.

For $n \geqslant 3$ the operator $\triangle_s$ has a system of eigenfunctions $Y^{(n)}_{l,\mathbf{m}}$, full in $\mathcal{L}^2\left(\mathbf{S}^{n-1}, d\mu_{\mathbf{S}^{n-1}}\right)$ [199]. Here $l = 0, 1, 2, \ldots,$ and $\mathbf{m}$ is a multiindex of the form

$$\mathbf{m} = (m_1, \ldots, m_{n-3}, \pm m_{n-2}),\ m_i \in \mathbb{Z},\ l \geqslant m_1 \geqslant \ldots \geqslant m_{n-2} \geqslant 0\ .$$

For $n = 2$ the full system of eigenfunctions for $\triangle_s$ is $Y_l = e^{\mathbf{i}l\phi}$, $l \in \mathbb{Z}$. The corresponding eigenvalues in all cases are $-l(l+n-2)$. The eigenfunction, corresponding to $l = 0$ is constant and is supposed to be 1.

Let $\psi \in \mathrm{Dom}(H_V)$ be an eigenfunction of H_V. It can be expanded in the form

$$\psi = \sum_{l,\mathbf{m}} \chi_{l,\mathbf{m}} Y^{(n)}_{l,\mathbf{m}},$$

where $\chi_{l,\mathbf{m}} \in W^{1,2}((0,1), d\mu_1)$ and then

$$H_V\psi = \sum_{l,\mathbf{m}}\left(-\frac{(1-v^2)^2}{2R^2}\left(\chi''_{l,\mathbf{m}}(v) + \left(\frac{n-1}{v} + \frac{(n-3)v}{1-v^2}\right)\chi'_{l,\mathbf{m}}(v)\right)\right.$$
$$\left. + \left(V + \frac{1-v^2}{2R^2v^2}l(l+n-2)\right)\chi_{l,\mathbf{m}}(v)\right)Y^{(n)}_{l,\mathbf{m}} = \sum_{l,\mathbf{m}} E\chi_{l,\mathbf{m}}(v)Y^{(n)}_{l,\mathbf{m}}\ .$$

Since functions $Y^{(n)}_{l,\mathbf{m}}$ are linearly independent, one gets the spectral problem for the radial component $\chi_{l,\mathbf{m}}(v) =: \chi_l(v)$ of the eigenfunction $\chi_{l,\mathbf{m}}Y^{(n)}_{l,\mathbf{m}}$

$$\begin{aligned}&\frac{(1-v^2)^2}{2R^2}\left(\chi_l''(v)+\left(\frac{n-1}{v}+\frac{(n-3)v}{1-v^2}\right)\chi_l'(v)\right)\\&+\left(E-V-\frac{1-v^2}{2R^2v^2}l(l+n-2)\right)\chi_l(v)=0,\ 0<v<1\,.\end{aligned}\tag{6.32}$$

The condition $\psi \in \mathrm{Dom}(H_V)$ implies the restrictions on asymptotics for $\chi_l(v)$ as $v \to +0$ and $v \to 1-0$.

The Coulomb Problem

Suppose $n \geqslant 2$, $V = V_c = -\dfrac{\gamma}{Rv}$. Now (6.32) looks like

$$\begin{aligned}&\chi_l''(v)+\left(\frac{n-1}{v}+\frac{(n-3)v}{1-v^2}\right)\chi_l'(v)\\&+\left(\frac{2ER^2}{(1-v^2)^2}-\frac{l(l+n-2)}{v^2(1-v^2)}+\frac{2R\gamma}{v(1-v^2)^2}\right)\chi_l(v)=0\,,0<v<1\,.\end{aligned}\tag{6.33}$$

Comparing it with (B.1) one concludes that the last equation in the complex plane is a Fuchsian one with singular points $\pm 1, 0$. The point ∞ is regular that can be verified by the substitution $v = \zeta^{-1}, \zeta \to 0$.

Calculations of the characteristic exponents at these points leads to the expressions:

$$\begin{aligned}\rho_1^{(1)} &= \frac{n-1+\sqrt{(n-1)^2-8R(ER+\gamma)}}{4}\,,\\ \rho_2^{(1)} &= \frac{n-1-\sqrt{(n-1)^2-8R(ER+\gamma)}}{4}\,,\\ \rho_1^{(-1)} &= \frac{n-1+\sqrt{(n-1)^2-8R(ER-\gamma)}}{4}\,,\\ \rho_2^{(-1)} &= \frac{n-1-\sqrt{(n-1)^2-8R(ER-\gamma)}}{4}\,,\\ \rho_1^{(0)} &= l,\ \rho_2^{(0)} = 2-l-n\,.\end{aligned}$$

Moreover, (6.33) is the Riemannian equation of the form (B.2) with $\rho_2^{(\pm 1)} \leqslant \rho_1^{(\pm 1)}, \rho_2^{(0)} \leqslant \rho_1^{(0)}$.

The condition $\chi_l Y_{l,\mathbf{m}}^{(n)} \in \mathrm{Dom}(H_{V_c})$ implies that

$$\int_0^1 \frac{\chi_l^2 v^{n-1} dv}{(1-v^2)^{\frac{n+1}{2}}} < \infty\,,\tag{6.34}$$

$$\triangle\left(\chi_l Y_{l,\mathbf{m}}^{(n)}\right) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbf{H}^n(\mathbb{R}), d\mu)\,.\tag{6.35}$$

Inequality (6.34) gives $\rho^{(0)} > -n/2$ and $\rho^{(1)} > (n-1)/4$, if $\chi_l \sim v^{\rho^{(0)}}$ as $v \to 0$ and $\chi_l \sim v^{\rho^{(1)}}$ as $v \to 1$. Since the inequality $\rho_2^{(0)} = 2-l-n > -n/2$ implies $n < 2(2-l)$, the asymptotic $\chi_l \sim v^{\rho_2^{(0)}}$ contradicts to $n \geqslant 2$ for $l \geqslant 1$. If

$l = 0$, then $n = 3$ or $n = 2$. For $n = 3, l = 0$ the asymptotic $\chi_0 \sim v^{\rho_2^{(0)}} = 1/v$ leads to $\triangle\chi_l(v) \sim \delta(0)$ that contradicts to (6.35). In the case $n = 2, l = 0$ it holds $\rho_1^{(0)} = \rho_2^{(0)} = 0$ and the theory of Fuchsian equations [50] implies that canonical asymptotics of a solution for (6.33) near $v = 0$ are 1 and $\log v$. The latter asymptotic again leads to $\triangle\chi_l(v) \sim \delta(0)$ that contradicts to (6.35). Thus, it should be $\chi_l \sim v^{\rho_1^{(0)}}$ as $v \to 0$.

The inequality

$$\rho_{1,2}^{(1)} = \frac{n-1 \pm \sqrt{(n-1)^2 - 8R(ER+\gamma)}}{4} > \frac{n-1}{4}$$

implies that it should be $\chi_l \sim (v-1)^{\rho_1^{(1)}}$ as $v \to 1$ and

$$(n-1)^2 - 8R(ER+\gamma) > 0 \; . \tag{6.36}$$

Conversely if $\chi_l \sim v^{\rho_1^{(0)}}$ as $v \to 0$, $\chi_l \sim (v-1)^{\rho_1^{(1)}}$ as $v \to 1$ and inequality (6.36) is valid, then $\chi_l Y_{l,\mathbf{m}}^{(n)} \in \mathrm{Dom}(H_{V_c})$.

According to the general theory (see appendix B) (6.33) can be reduced to the hypergeometric equation by the substitution

$$z := \frac{v-1}{2v}, \; w_l(z) := \chi_l(v) \left(\frac{v}{1-v}\right)^{\rho_1^{(1)}} \left(\frac{v}{1+v}\right)^{\rho_2^{(-1)}} .$$

By expressions

$$\left(\frac{v}{1+v}\right)^{\rho_2^{(-1)}} \quad \text{and} \quad \left(\frac{v}{1+v}\right)^{\rho_1^{(1)}}$$

we mean here branches of multifunctions, which are holomorphic on the expanded complex plane $\mathbb{C} \cup \infty$ with the cut along $[-\infty, 0] \cup [1, +\infty]$ and are real on the segment $[0, 1]$.

This substitution moves the singular points $0, 1, -1$ respectively into $\infty, 0, 1$ and the real interval $(0, 1)$ into $(-\infty, 0)$. The function $w_l(z)$ satisfies (B.5) and its characteristic exponents are as follows: at the point $z = \infty$ they are

$$\begin{aligned}\alpha = \rho_1^{(1)} + \rho_2^{(-1)} + \rho_1^{(0)} &= \frac{n-1}{2} + l \\ &+ \frac{1}{4}\left(\sqrt{(n-1)^2 - 8R(ER+\gamma)} - \sqrt{(n-1)^2 - 8R(ER-\gamma)}\right), \\ \beta = \rho_1^{(1)} + \rho_2^{(-1)} + \rho_2^{(0)} &= \frac{3-n}{2} - l \\ &+ \frac{1}{4}\left(\sqrt{(n-1)^2 - 8R(ER+\gamma)} - \sqrt{(n-1)^2 - 8R(ER-\gamma)}\right) < \alpha \; ;\end{aligned}$$

at the point $z = 0$ they are 0 and

$$1 - \widetilde{\gamma} := \rho_2^{(1)} - \rho_1^{(1)} = -\frac{1}{2}\sqrt{(n-1)^2 - 8R(ER+\gamma)} \; ;$$

and at the point $z = 1$ they are 0 and $\widetilde{\gamma}-\alpha-\beta$. Note that $\alpha-\beta = n+2l-2 =: l_1 \geqslant 1$.

The function $w_l(z)$ for $-\infty < z < 0$ corresponds to the function $\chi_l(v)$ for $0 < v < 1$ and should has asymptotics $w_l(z) \sim z^{-\alpha}$ as $z \to -\infty$ and $w_l(z) \sim$ const as $z \to 0$. According to appendix B it means that

$$w_l(z) = F(\alpha, \beta; \widetilde{\gamma}; z) \quad \text{and} \quad \lim_{z\to-\infty} F(\alpha, \beta; \widetilde{\gamma}; z) z^{\beta} = 0 \, .$$

Due to (B.13) for $\alpha - \beta = l_1 \in \mathbb{N}$ it holds

$$\lim_{z\to-\infty} F(\alpha, \beta; \widetilde{\gamma}; z)(-z)^{\beta} = \frac{\Gamma(\widetilde{\gamma})\Gamma(l_1)}{\Gamma(\widetilde{\gamma}-\beta)\Gamma(\alpha)} \, . \tag{6.37}$$

The gamma-function has no zeros in $\mathbb{C}$ and has poles at the points $0, -1, -2, \ldots$. Since $\widetilde{\gamma} > 0$ and

$$\widetilde{\gamma} - \beta = \frac{n-1}{2} + l + \frac{1}{4}\Big(\sqrt{(n-1)^2 - 8R(ER+\gamma)} + \sqrt{(n-1)^2 - 8R(ER-\gamma)}\Big) > 0,$$

it should be $\alpha = -k+1$, $k \in \mathbb{N}$ or equivalently

$$\sqrt{(n-1)^2 - 8R(ER+\gamma)} - \sqrt{(n-1)^2 - 8R(ER-\gamma)} = -4(k+l) - 2n + 6 \, .$$

Multiplying the last equation by

$$\sqrt{(n-1)^2 - 8R(ER+\gamma)} + \sqrt{(n-1)^2 - 8R(ER-\gamma)}$$

one gets

$$\sqrt{(n-1)^2 - 8R(ER+\gamma)} + \sqrt{(n-1)^2 - 8R(ER-\gamma)} = \frac{8R\gamma}{2(k+l)+n-3} \, .$$

Taking into account (6.36), one gets

$$\sqrt{(n-1)^2 - 8R(ER+\gamma)} = \frac{4R\gamma}{2(k+l)+n-3} - (2(k+l)+n-3) > 0 \, .$$

This leads to the final formula

$$E_{k,l} = \frac{(n-1)^2}{8R^2} - \frac{2\gamma^2}{(2(k+l)+n-3)^2} - \frac{(2(k+l)+n-3)^2}{8R^2}, \quad 1 \leqslant k < \sqrt{R\gamma} + \frac{3-n}{2} - l \, . \tag{6.38}$$

For $\alpha = -k+1$, $k \in \mathbb{N}$ the function $F(\alpha, \beta; \widetilde{\gamma}; z)$ is a polynomial of $(k-1)$th degree and the radial part of the eigenfunction has the form:

$$\chi_{k,l}(v) = \left(\frac{1-v}{v}\right)^{\rho_1^{(1)}} \left(\frac{1+v}{v}\right)^{\rho_2^{(-1)}} \sum_{i=0}^{k-1} \frac{(1-k)_i(3-k-n-2l)_i}{i!(\widetilde{\gamma})_i} \left(\frac{v-1}{2v}\right)^i , \tag{6.39}$$

where

$$\widetilde{\gamma} = \frac{5-n}{2} - k - l + \frac{2R\gamma}{2(k+l)+n-3}\ ,\ \rho_1^{(1)} = \frac{1-k-l}{2} + \frac{R\gamma}{2(k+l)+n-3}\ ,$$
$$\rho_2^{(-1)} = \frac{1-k-l}{2} - \frac{R\gamma}{2(k+l)+n-3}\ .$$

Let us make some remarks, concerning formulae (6.38) and (6.39). In the limit $R \to \infty$ formula (6.38) gives the energy levels for the Coulomb problem in Euclidean space

$$E_{k,l} = -\frac{2\gamma^2}{(2(k+l)+n-3)^2},\ k \in \mathbb{N}\ .$$

For $n = 3$ it is the *Balmer formula*. Eigenfunctions (6.39) also converge to their Euclidean counterparts in the limit $R \to \infty$.

Note that if the potential V_c is normalized in accordance with Remark 6.1, then there should be the additional constant summand γ/R in (6.38).

As in Euclidean case the energy levels of bound states depend only on the sum of $k + l$ that means the similar degeneracy of energy levels.

Contrary to Euclidean case there are here only finite number of bound states; for $\sqrt{R\gamma} + (3-n)/2 - l < 1$ there are no bound states at all.

The values $E_{k,l}$ increase with the grow of k, while the inequality in (6.38) is valid. If $\sqrt{R\gamma} + (3-n)/2 - l = k_1 + \varepsilon$, $k_1 \in \mathbb{N}, \varepsilon > 0$, then it holds

$$E_{k_1,l} \to \frac{(n-1)^2}{8R^2} - \frac{\gamma}{R}, \quad \text{as} \quad \varepsilon \to +0$$

that exceeds the classical energy threshold (see (6.9)) of the finite motion by the term

$$\frac{(n-1)^2}{8R^2}\ . \tag{6.40}$$

This term in (6.38) has the following nature. The spectrum of the Laplace-Beltrami operator on $\mathcal{L}^2(\mathbf{H}^n(\mathbb{R}), d\mu)$ (purely continues) is

$$\left(-\infty, -\frac{(n-1)^2R^2}{4}\right]$$

(see [119]) and therefore the summand $(n-1)^2/(8R^2)$ is simply the lower bound of the spectrum for the operator $-\frac{1}{2}\triangle$. For $n = 3$ it was mentioned already in [73]. Note that some authors prefer to exclude this term from (6.38) by the subtracting it from the Hamiltonian H_V.

The Oscillator Problem

Suppose $n \geqslant 2$, $V = V_o = \dfrac{1}{2}\omega^2R^2v^2$. Equation (6.32) takes now the form

$$\chi_l''(v) + \left(\frac{n-1}{v} + \frac{(n-3)v}{1-v^2}\right)\chi_l'(v) + \left(\frac{2ER^2}{(1-v^2)^2} - \frac{l(l+n-2)}{v^2(1-v^2)} - \frac{R^4\omega^2 v^2}{(1-v^2)^2}\right)\chi_l(v) = 0 \quad 0 < v < 1\,. \tag{6.41}$$

In the complex plane this equation is a Fuchsian one with singular points $\pm 1, 0$ and ∞. Characteristic exponents at these points are:

$$\rho_1^{(1)} = \rho_1^{(-1)} = \frac{n-1+\sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2}}{4},\ \rho_1^{(\infty)} = \frac{1+\sqrt{1+4R^4\omega^2}}{2},$$
$$\rho_2^{(1)} = \rho_2^{(-1)} = \frac{n-1-\sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2}}{4},\ \rho_2^{(\infty)} = \frac{1-\sqrt{1+4R^4\omega^2}}{2},$$
$$\rho_1^{(0)} = l,\ \rho_2^{(0)} = 2 - l - n\,.$$

The similar considerations as for the Coulomb problem above imply that $\chi_l Y_{l,\mathbf{m}}^{(n)} \in \mathrm{Dom}(H_{V_o})$ iff $\chi_l \sim v^{\rho_1^{(0)}}$ as $v \to 0$; $\chi_l \sim (v-1)^{\rho_1^{(1)}}$ as $v \to 1$; and

$$(n-1)^2 - 8ER^2 + 4R^4\omega^2 > 0\,. \tag{6.42}$$

One can reduce (6.41) to the Riemannian equation using the equalities $\rho_i^{(1)} = \rho_i^{(-1)}$, $i = 1, 2$. The change of variables[3]

$$\chi_l(v) = w_l(z) z^{\frac{1}{2}\rho_1^{(0)}} (1-z)^{\rho_1^{(1)}},\ z = v^2$$

glues the points ± 1 together and leads directly to the hypergeometric equation (B.5) for the function $w_l(z)$ with

$$\alpha = \frac{1}{2}\left(\rho_1^{(\infty)} + l\right) + \rho_1^{(1)} = \frac{1}{4}\left(n + 2l + \sqrt{1+4R^4\omega^2} + \sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2}\right),$$
$$\beta = \frac{1}{2}\left(\rho_2^{(\infty)} + l\right) + \rho_1^{(1)} = \frac{1}{4}\left(n + 2l - \sqrt{1+4R^4\omega^2} + \sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2}\right),$$
$$\gamma = 1 + \frac{1}{2}(l - \rho_2^{(0)}) = l + \frac{n}{2} > 0\,.$$

Here by $z^{\frac{1}{2}\rho_1^{(0)}}(1-z)^{\rho_1^{(1)}}$ we mean the branch of a multifunction, holomorphic on $\mathbb{C} \cup \infty$ with the cut along $[-\infty, 0] \cup [1, +\infty]$ and real on the interval $(0, 1)$.

The function $w_l(z)$ on the interval $(0, 1)$ should be ~ 1 as $z \to +0$ or $z \to 1 - 0$. Taking into account the inequality $1 - \gamma = 1 - l - n/2 < 0$ and formula (B.10), one gets

$$w_l(z) = F(\alpha, \beta; \gamma; z) \quad \text{and}$$
$$\lim_{z \to 1-0} F(\alpha, \beta; \gamma; z)(1-z)^{\alpha+\beta-\gamma} = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\alpha)\Gamma(\beta)} = 0\,.$$

[3] This change corresponds to the first case of Theorem B.1.

Since $\alpha > 0$, the only possibility is $\beta = -k$, $k = 0, 1, 2, \dots$ or equivalently

$$\sqrt{1 + 4R^4\omega^2} - \sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2} = 2(2k + l) + n \ .$$

Multiplying the last equation by

$$\sqrt{1 + 4R^4\omega^2} + \sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2},$$

one obtains

$$\sqrt{1 + 4R^4\omega^2} + \sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2} = \frac{8ER^2 + 1 - (n-1)^2}{2(2k + l) + n} \ .$$

This gives

$$2\sqrt{1 + 4R^4\omega^2} = \frac{8ER^2 + 1 - (n-1)^2}{2(2k + l) + n} + 2(2k + l) + n,$$
$$2\sqrt{(n-1)^2 - 8ER^2 + 4R^4\omega^2} = \frac{8ER^2 + 1 - (n-1)^2}{2(2k + l) + n} - 2(2k + l) - n > 0 \ .$$

Finally it holds

$$E_{k,l} = \omega(2k + l + \frac{n}{2})\sqrt{1 + \frac{1}{4R^4\omega^2}} - \frac{1}{2R^2}\left((2k + l)^2 + (2k + l)n + \frac{n}{2}\right), \quad l \leqslant 2k + l < \omega R^2\sqrt{1 + \frac{1}{4R^4\omega^2}} - \frac{n}{2} \ . \tag{6.43}$$

The corresponding radial parts of eigenfunctions has the form:

$$\chi_{k,l}(v) = v^l\left(1 - v^2\right)^{\rho_1^{(1)}} \sum_{i=0}^{k} \frac{(\alpha)_i(-k)_i}{(l + n/2)_i}\frac{v^i}{i!},$$

where $\alpha, \rho_1^{(1)}$ are defined by formulae above for $E = E_{k,l}$.

Remarks above, concerning formulae (6.38) and (6.39) are valid also in this case. Note only that if

$$\omega R^2\sqrt{1 + \frac{1}{4R^4\omega^2}} - \frac{n}{2} = 2k_1 + l + \varepsilon, \ \varepsilon > 0, k_1 = 0, 1, 2. \dots,$$

then

$$E_{k_1,l} \to \frac{(n-1)^2}{8R^2} + \frac{\omega^2R^2}{2} \quad \text{as} \quad \varepsilon \to +0$$

that again exceeds the classical energy threshold of the finite motion by term (6.40) (see (6.25)).

6.3.2 The Spherical Case

Calculations here are similar to the hyperbolic case, but contain some subtleties connected with a behavior of eigenfunctions in the complex plane. We shall use here the model (1.20), which corresponds to the Laplace-Beltrami operator on $\mathbf{S}^n$ in the form

$$\triangle = \frac{(1+v^2)^2}{R^2}\left(\frac{\partial^2}{\partial v^2} + \left(\frac{n-1}{v} - \frac{(n-3)v}{1+v^2}\right)\frac{\partial}{\partial v}\right) + \frac{1+v^2}{R^2 v^2}\triangle_s,\ v \in \mathbb{R}\ , \tag{6.44}$$

where $\triangle_s$ is the same as in (6.31). The limits $v \to \pm 0$ correspond to the poles of $\mathbf{S}^n$ and the limit $v \to \infty$ corresponds to the equator defined by $r = 1$ in model (1.19).

The Coulomb Problem

Theorem 2.11 implies here the self-adjointness of H_{V_c} for $n \geqslant 2$, where $V = V_c = -\dfrac{\gamma}{Rv}$ with $\mathrm{Dom}(H_V)$ given by (2.28). Here $V_1 = V_c$ if $v < 0$ and $V = 0$ otherwise. Below we clarify the calculations from [185] and generalize them for the case $n \neq 3$.

Similarly to (6.33) one obtains a Fuchsian equation for the radial part $\chi_l(v)$ of the eigenfunction $\chi_l Y^{(n)}_{l,\mathbf{m}}$ in the form:

$$\begin{aligned}&\chi_l''(v) + \left(\frac{n-1}{v} - \frac{(n-3)v}{1+v^2}\right)\chi_l'(v)\\ &\quad + \left(\frac{2ER^2}{(1+v^2)^2} - \frac{l(l+n-2)}{v^2(1+v^2)} + \frac{2R\gamma}{v(1+v^2)^2}\right)\chi_l(v) = 0, v \in \mathbb{R}\ .\end{aligned} \tag{6.45}$$

Singular points for this equation in the complex plane are $0, \pm\mathbf{i}$ (∞ is a regular point as for (6.33)) with characteristic exponents:

$$\begin{aligned}\rho_1^{(\mathbf{i})} &= \frac{1}{4}(n-1+s),\ \rho_2^{(\mathbf{i})} = \frac{1}{4}(n-1-s),\ \rho_1^{(-\mathbf{i})} = \frac{1}{4}(n-1+\bar{s}),\\ \rho_2^{(-\mathbf{i})} &= \frac{1}{4}(n-1-\bar{s}),\ \rho_1^{(0)} = l,\ \rho_2^{(0)} = 2-l-n,\end{aligned}$$

where s and $-s$ are roots of the equation $s^2 = (n-1)^2 + 8ER^2 - 8\mathbf{i}R\gamma$. The main difference from the eigenvalue problems, considered above for the hyperbolic case, is that here only one singular point lies on the interval (namely $0 \in \mathbb{R} \cup \infty$) under consideration. This situation arose in result of confluence on the v-plane of two poles on the sphere $\mathbf{S}^n$.

As in the hyperbolic case above the function $\chi_l(v)$ should be $\sim c_+ v^{\rho_1^{(0)}} = c_+ v^l$, $c_+ = \mathrm{const}$ as $v \to +0$, $\mathbb{R} \ni v > 0$, otherwise it would be $\chi_l Y^{(n)}_{l,\mathbf{m}} \notin \mathrm{Dom}(H_{V_c})$. At the same time it should be also $\chi_l \sim c_- v^l$, $c_- = \mathrm{const}$ as $v \to -0$, $\mathbb{R} \ni v < 0$, since the limit $v = 2r/(1-r^2) = \tan(\rho/R) \to -0$ corresponds to another pole $r \to +\infty$ (or equivalently $\rho = \pi R$) of the sphere $\mathbf{S}^n$.

One can reduce (6.45) to hypergeometric equation (B.5) for a function $w(z)$ with parameters $\alpha, \beta, \widetilde{\gamma}$ by the substitution:

$$z := \frac{2v}{v+\mathbf{i}},\ w_l(z) := \chi_l(v)\left(\frac{v+\mathbf{i}}{v}\right)^l \left(\frac{v+\mathbf{i}}{v-\mathbf{i}}\right)^{\rho_2^{(\mathbf{i})}}, \tag{6.46}$$

which moves the triple of singular points $v = 0, \mathbf{i}, -\mathbf{i}$ into the triple $z = 0, 1, \infty$. By the expression

$$\left(\frac{v+\mathbf{i}}{v-\mathbf{i}}\right)^{\rho_2^{(\mathbf{i})}} \tag{6.47}$$

we mean here the branch of the multifunction, that is holomorphic on $(\mathbb{C} \cup \infty)\backslash[-\mathbf{i}, \mathbf{i}]$ and equals 1 at $v = \infty$. Since this branch is holomorphic at $v = \infty$, the function $w_l(z)$ is holomorphic at $z = 2$.

Note that the real line $\mathbb{R}$ on the v-plane is mapped by substitution (6.46) into the circle $\mathbb{S}^1$ on the z-plane, defined by the equation $|z-1| = 1$.

It is easily seen that

$$\alpha = \rho_2^{(-\mathbf{i})} + l + \rho_2^{(\mathbf{i})} = \frac{n-1}{2} + l - \frac{1}{4}(\bar{s} + s) \in \mathbb{R},$$
$$\beta = \rho_1^{(-\mathbf{i})} + l + \rho_2^{(\mathbf{i})} = \frac{n-1}{2} + l + \frac{1}{4}(\bar{s} - s) \notin \mathbb{R},\ \widetilde{\gamma} = 1 - \rho_2^{(0)} + l = n + 2l - 1\,.$$

Since $\chi_l \sim c_\pm v^l$ as $v \to \pm 0$, $v \in \mathbb{R}$, the function $w_l(z)$ on $\mathbb{S}^1$ is bounded near the point $z = 0$ and $1 - \widetilde{\gamma} \leqslant 0$, it holds (see (B.6))

$$w_l(z) = w_{+,l}(z) := \widetilde{c}_+ F(\alpha, \beta; \widetilde{\gamma}; z),\ z \in \mathbb{S}^1_+ := \left(z \in \mathbb{S}^1, \operatorname{Im} z > 0\right),\ \widetilde{c}_+ = \text{const},$$
$$w_l(z) = w_{-,l}(z) := \widetilde{c}_- F(\alpha, \beta; \widetilde{\gamma}; z),\ z \in \mathbb{S}^1_- := \left(z \in \mathbb{S}^1, \operatorname{Im} z < 0\right),\ \widetilde{c}_- = \text{const}\,.$$

Note that in [185] Stevenson made an assumption equivalent to $\widetilde{c}_+ = \widetilde{c}_-$ without any proof.

Functions $w_{\pm,l}(z)$ should be analytic continuations of each other through the point $z = 2$.[4] Due to formula (B.7) (applicable since $\widetilde{\gamma} - \alpha - \beta \notin \mathbb{R}$) it means that functions

$$\widetilde{c}_+ \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma} - \alpha - \beta)}{\Gamma(\widetilde{\gamma} - \alpha)\Gamma(\widetilde{\gamma} - \beta)} F(\alpha, \beta; \alpha + \beta - \widetilde{\gamma} + 1; 1 - z),\ z \in \mathbb{S}^1_+$$

and

$$\widetilde{c}_- \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma} - \alpha - \beta)}{\Gamma(\widetilde{\gamma} - \alpha)\Gamma(\widetilde{\gamma} - \beta)} F(\alpha, \beta; \alpha + \beta - \widetilde{\gamma} + 1; 1 - z),\ z \in \mathbb{S}^1_-$$

are analytic continuations of each other through the point $z = 2$ as well as functions

$$\widetilde{c}_+ \frac{\Gamma(\widetilde{\gamma})\Gamma(\alpha + \beta - \widetilde{\gamma})}{\Gamma(\alpha)\Gamma(\beta)} (1 - z)^{\widetilde{\gamma} - \alpha - \beta} F(\widetilde{\gamma} - \alpha, \widetilde{\gamma} - \beta; \widetilde{\gamma} - \alpha - \beta + 1; 1 - z),\ z \in \mathbb{S}^1_+$$

and

[4] Recall that the function $F(\alpha', \beta'; \gamma'; z)$ is holomorphic in $\mathbb{C}\backslash[1, +\infty)$.

$$\widetilde{c}_- \frac{\Gamma(\widetilde{\gamma})\Gamma(\alpha+\beta-\widetilde{\gamma})}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\widetilde{\gamma}-\alpha-\beta}F(\widetilde{\gamma}-\alpha,\widetilde{\gamma}-\beta;\widetilde{\gamma}-\alpha-\beta+1;1-z),\ z\in\mathbb{S}^1_- \ .$$

These requirements are equivalent to the system

$$\begin{aligned}(\widetilde{c}_+ - \widetilde{c}_-)\frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma}-\alpha-\beta)}{\Gamma(\widetilde{\gamma}-\alpha)\Gamma(\widetilde{\gamma}-\beta)} &= 0,\\ (\widetilde{c}_+ - \widetilde{c}_- \exp(2\pi\mathbf{i}(\widetilde{\gamma}-\alpha-\beta)))\frac{\Gamma(\widetilde{\gamma})\Gamma(\alpha+\beta-\widetilde{\gamma})}{\Gamma(\alpha)\Gamma(\beta)} &= 0\ .\end{aligned} \tag{6.48}$$

Since $\widetilde{\gamma}-\alpha-\beta \notin \mathbb{R}$, linear system (6.48) has a nontrivial solution $\widetilde{c}_+, \widetilde{c}_-$ iff either

$$\frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma}-\alpha-\beta)}{\Gamma(\widetilde{\gamma}-\alpha)\Gamma(\widetilde{\gamma}-\beta)} = 0 \quad \text{or} \quad \frac{\Gamma(\widetilde{\gamma})\Gamma(\alpha+\beta-\widetilde{\gamma})}{\Gamma(\alpha)\Gamma(\beta)} = 0\ .$$

Taking into account that $\widetilde{\gamma}-\beta \notin \mathbb{R}$, $\beta \notin \mathbb{R}$, one gets $\widetilde{\gamma}-\alpha = -k+1$ or $\alpha = -k+1$, $k\in\mathbb{N}$ and thus

$$s+\bar{s} = \pm\left(4(k+l)+2n-6\right)\ . \tag{6.49}$$

Without loss of generality suppose that $s+\bar{s}>0$ and therefore $\alpha=-k+1$, $k\in\mathbb{N}$, $\widetilde{c}_+ = \widetilde{c}_-$.[5]

The similar calculations as in the hyperbolic case yield the expression

$$s = 2(k+l)+n-3-\frac{4\mathbf{i}R\gamma}{2(k+l)+n-3}$$

and finally the energy levels

$$E_{k,l} = -\frac{(n-1)^2}{8R^2}-\frac{2\gamma^2}{(2(k+l)+n-3)^2}+\frac{(2(k+l)+n-3)^2}{8R^2},\ k\in\mathbb{N}\ . \tag{6.50}$$

Here there are infinite number of energy levels and $E_{k,l}\to+\infty$ as $k\to+\infty$ in a monotone way. As in the hyperbolic case in the limit $R\to\infty$ formula (6.50) describes energy levels of Coulomb problem in Euclidean space. In the limit $\gamma\to 0$ (6.50) becomes the formula for energy levels of the free motion on the sphere.

The eigenfunctions, corresponding to (6.50), are:

$$\chi_{k,l}(v) = \left(\frac{v}{v+\mathbf{i}}\right)^l \left(\frac{v-\mathbf{i}}{v+\mathbf{i}}\right)^{\rho_2^{(\mathbf{i})}} \sum_{p=0}^{k-1}\frac{(1-k)_p(\beta)_p}{p!(n+2l-1)_p}\left(\frac{2v}{v+\mathbf{i}}\right)^p,$$

where β and $\rho_2^{(\mathbf{i})}$ are given by

$$\beta = \frac{n-1}{2}+l+\frac{2\mathbf{i}R\gamma}{2(k+l)+n-3},\ \rho_2^{(\mathbf{i})} = \frac{1-k-l}{2}+\frac{\mathbf{i}R\gamma}{2(k+l)+n-3}\ .$$

[5] Both choices of a sign for $s+\bar{s}$ in (6.49) lead to the same energy levels and eigenfunctions, but expressions for eigenfunctions can have different forms.

One can also express eigenfunctions through the more convenient coordinate ρ (see (1.21)):

$$\chi_{k,l}(\rho) = \exp\left(2\mathbf{i}\rho\rho_2^{(\mathbf{i})}/R\right)\sum_{p=0}^{k-1}\frac{2^p(1-k)_p(\beta)_p}{p!(n+2l-1)_p\mathbf{i}^p}$$
$$\times \sin^{l+p}(\rho/R)\exp\left(\mathbf{i}\rho(l+p)/R\right),\ 0\leqslant\rho\leqslant\pi\ .$$

Note that function (6.47) becomes $-\exp\left(2\mathbf{i}\rho\rho_2^{(\mathbf{i})}/R\right)$, which is continuous at $\rho=\pi/2$ ($v=\infty$) and has different values at the points $\rho=0,\pi R$ ($v=\pm 0$) as was stated above.

The Oscillator Problem

The oscillator potential $V_o=\frac{1}{2}R^2\omega^2\tan^2(\rho/R)$ for the sphere $\mathbf{S}^n$, $n\geqslant 2$ does not belong to the space $\mathcal{L}^1_{\mathrm{loc}}(\mathbf{S}^n,d\mu)$, since $V_o\to+\infty$ as $\rho\to\pi R$. Therefore, we shall use Theorem 2.12 for constructing the self-adjoint extension $(H_{V_o})_F$ of the operator H_{V_o} for which the set M' coincides with the hemisphere defined by the inequality $\rho<\pi R$. The domain of this extension is given by formula (2.32).

This definition of H_{V_o} corresponds to the physical idea that infinite potential barrier implies the Dirichlet condition for a wave function on the boundary $\mathbf{S}^{n-1}$ of M', defined by the equation $\rho=\pi R$.

Now the equation for the radial part $\chi_l(v)$ of an eigenfunction $\chi_l(v)Y^n_{l,\mathbf{m}}$ takes the form

$$\chi_l''(v)+\left(\frac{n-1}{v}-\frac{(n-3)v}{1+v^2}\right)\chi_l'(v)$$
$$+\left(\frac{2ER^2}{(1+v^2)^2}-\frac{l(l+n-2)}{v^2(1+v^2)}-\frac{R^4\omega^2v^2}{(1+v^2)^2}\right)\chi_l(v)=0,\ 0<v<+\infty\ . \tag{6.51}$$

In the complex plane this equation is a Fuchsian one with singular points $\pm\mathbf{i},0$ and ∞. Its characteristic exponents at these points are:

$$\rho_1^{(\mathbf{i})}=\rho_1^{(-\mathbf{i})}=\frac{n-1+\sqrt{(n-1)^2+8ER^2+4R^4\omega^2}}{4},$$
$$\rho_1^{(\infty)}=\frac{1+\sqrt{1+4R^4\omega^2}}{2},$$
$$\rho_2^{(\mathbf{i})}=\rho_2^{(-\mathbf{i})}=\frac{n-1-\sqrt{(n-1)^2+8ER^2+4R^4\omega^2}}{4},$$
$$\rho_2^{(\infty)}=\frac{1-\sqrt{1+4R^4\omega^2}}{2},$$
$$\rho_1^{(0)}=l,\ \rho_2^{(0)}=2-l-n\ .$$

The similar arguments as for the oscillator potential in the hyperbolic space lead to the asymptotic $\chi_l\sim v^l$ as $v\to+0$. On the other hand, the condition $\chi_l(v)Y^n_{l,\mathbf{m}}\in\mathrm{Dom}(q_{-\triangle_F+\mathrm{id}})$ implies the convergence of the integral

$$\int_{M'} g\left(\operatorname{grad}\left(\chi_l Y^n_{l,\mathbf{m}}\right), \operatorname{grad}\left(\chi_l Y^n_{l,\mathbf{m}}\right)\right) d\mu$$

that leads to

$$\int_0^{+\infty} \frac{\left((1+v^2)\chi_l'\right)^2 v^{n-1} dv}{(1+v^2)^{\frac{n+1}{2}}} < \infty.$$

The convergence of the latter integral as $v \to +\infty$ implies $\rho^{(\infty)} > 1/2$, if $\chi_l \sim v^{-\rho^{(\infty)}}$ as $v \to +\infty$. Thus, it should be $\chi_l \sim v^{-\rho_1^{(\infty)}}$ as $v \to +\infty$.

Conversely, it can be easily verified that for a solution of (6.51) with asymptotics $\chi_l \sim v^l$ as $v \to +0$ and $\chi_l \sim v^{-\rho_1^{(\infty)}}$ as $v \to +\infty$ it holds $\chi_l(v) Y^n_{l,\mathbf{m}} \in \operatorname{Dom}\left((H_{V_o})_F\right)$.

The change of variables[6]

$$\chi_l(v) = w_l(z)(-z)^{l/2}(1-z)^{\rho_1^{(\mathbf{i})}},\ z = -v^2$$

leads to the hypergeometric equation (B.5) for the function $w_l(z)$ with

$$\begin{aligned}
\alpha &= \frac{1}{2}\left(\rho_1^{(\infty)} + l\right) + \rho_1^{(\mathbf{i})} \\
&= \frac{1}{4}\left(n + 2l + \sqrt{1 + 4R^4\omega^2} + \sqrt{(n-1)^2 + 8ER^2 + 4R^4\omega^2}\right), \\
\beta &= \frac{1}{2}\left(\rho_2^{(\infty)} + l\right) + \rho_1^{(\mathbf{i})} \\
&= \frac{1}{4}\left(n + 2l - \sqrt{1 + 4R^4\omega^2} + \sqrt{(n-1)^2 + 8ER^2 + 4R^4\omega^2}\right), \\
\gamma &= 1 + \frac{1}{2}(l - \rho_2^{(0)}) = l + \frac{n}{2} > 0\,.
\end{aligned}$$

Here the expression $(-z)^{l/2}(1-z)^{\rho_1^{(1)}}$ denotes the branch of the multifunction, holomorphic on $\mathbb{C} \cup \infty$ with the cut along $[0, +\infty]$ and real on $[-\infty, 0]$.

The half-line $v \in [0, +\infty]$ is transformed into the half-line $z \in [-\infty, 0]$ and asymptotics of $w_l(z)$ should be ~ 1 as $z \to -0$ and $\sim z^{-\alpha}$ as $z \to -\infty$. Since $\alpha > \beta$, one gets

$$w_l(z) = F(\alpha, \beta; \gamma; z) \quad \text{and} \quad \lim_{z \to -\infty} F(\alpha, \beta; \gamma; z)(-z)^\beta = \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} = 0,$$

due to (B.13). Therefore, it holds $\gamma - \beta = -k$, $k = 0, 1, 2, \ldots$ that implies

$$\sqrt{(n-1)^2 + 8ER^2 + 4R^4\omega^2} - \sqrt{1 + 4R^4\omega^2} = 2(2k + l) + n\,.$$

Calculations, similar to the hyperbolic case, give

$$E_{k,l} = \omega\left(2k + l + \frac{n}{2}\right)\sqrt{1 + \frac{1}{4R^4\omega^2}} + \frac{1}{2R^2}\left((2k+l)^2 + (2k+l)n + \frac{n}{2}\right), \quad k = 0, 1, 2, \ldots \tag{6.52}$$

[6] This change again corresponds to the first case of Theorem B.1.

In the limit $R \to \infty$ formula (6.52) becomes the well-known formula for the oscillator problem in Euclidean space. On the other hand, in the limit $\omega \to 0$ formula (6.52) describes energy levels of the free motion on the sphere.

The corresponding radial parts of eigenfunctions has the form:

$$\chi_{k,l}(v) = v^l \left(1+v^2\right)^{\rho_1^{(\mathbf{i})}} \sum_{i=0}^{\infty} \frac{(\alpha)_i (l+k+n/2)_i}{(l+n/2)_i} \frac{v^i}{i!},$$

where parameters $\alpha, \rho_1^{(\mathbf{i})}$ are defined by formulae above for $E = E_{k,l}$.

6.4 The History of the Problem of One and Two Particles in a Central Field on Constant Curvature Spaces

In this section we consider in the chronological order the history of a study of a one-body motion in central potentials in simply connected constant curvature spaces and the two-body problem with central interaction in the same spaces. Some papers, concerning the motion of a rigid body in these spaces, are mentioned in Sect. 7.4.1.

Due to a great quantity of scientific papers in most cases one can only declare that some paper has no priority in some (not widely known) question. A converse declaration is only a conjecture. The reader of this section should bear in mind this remark. At any case the history here is much more complete than in papers mentioned below. The history before the beginning of 20th century is difficult for study due to the absence of modern standards for scientific papers at that time.

The analogue of Newton (or Coulomb) force for the space $\mathbf{H}^3(\mathbb{R})$ was proposed already by founders of the hyperbolic geometry Lobachevski (in 1835–38) ([110], p. 159) and Bolyai (between 1848 and 1851) ([26], p. 156) as the value $F(\rho)$ that is inverse to the area of the sphere in $\mathbf{H}^3(\mathbb{R})$ of radius ρ with an attractive body in the center.

According to the footnote in [108] (1872, p. 117) Dirichlet had considered this force "schon früher".[7] The similar information is in the Schering paper [155] (1873, p. 149): "Mit diesem Gegenstande hat auch Dirichlet, wie ich jetzt erfahren, in der letzten Zeit seines Aufenthalts in Berlin sich beschäftigt; er hat darüber mit seinen Freunden gesprochen ohne von den Resultaten seiner Untersuchungen Mittheilung zu machen".

The analytical expression for the Newtonian potential in the space $\mathbf{H}^3(\mathbb{R})$ was written in 1870 by Schering [154] (see also his paper [155] of 1873), without any motivation and references to Lobachevski and Bolyai.

In 1873 Lipschitz considered a one-body motion in a central potential on the sphere $\mathbf{S}^2$ [109]. Although he knew that the central potential V_c satisfies the Laplace equation on $\mathbf{S}^3$, due to some reason he preferred to consider

[7] P.G. Dirichlet died in 1859.

another central potential $V(\rho) \sim \sin^{-1}(\rho/R)$ (in notations of Sect. 6.2.1). He found the general solution of this problem through elliptic functions.

In 1885 Killing found the generalization of all three Kepler laws for the sphere $\mathbf{S}^3$ [87]. He considered the attractive force as an inverse area of a 2-dimensional sphere in $\mathbf{S}^3$ as Lobachevski and Bolyai did before. In the next year these results were published also by Neumann in [129]. The expansion of these results onto the hyperbolic case was carried out in the Liebmann paper [103] in 1902 and later in 1905 in his book on noneuclidean geometry [105]. Note that he started from ellipses in $\mathbf{S}^3$ or $\mathbf{H}^3(\mathbb{R})$ and derived a potential in such a way that the first Kepler law would be valid. He derived also the generalization of the oscillator potential for these spaces from the requirement that a particle motion occurs along an ellipse, with its center coinciding with the center of the potential.

Also, in the same paper [87] Killing proved the variable separation in the two-centre Kepler problem on the sphere $\mathbf{S}^n$, which implies the integrability of this problem.

The generalization of the Bertrand theorem for spaces $\mathbf{S}^2$ and $\mathbf{H}^2(\mathbb{R})$ was proved by Liebmann in 1903 [104]. In the same year Stäckel wrote without any references ([182], p. 476): "...von Interesse ist auch, daß der bekannte Satz von Bertrand sein Analogon im absoluten Raume hat. Freilich sind das alles, um einen Ausdruck von Felix Klein zu gebrauchen, "selbst geschaffene Schmerzen", denn weder die Beobachtung an Planeten noch, was mehr sagen will, an Fixsternen nötigen uns, die altbewährte Euklidische Geometrie durch eine "Astralgeometrie" zu ersetzen". Similar words are also in the first edition (1905) of the Liebmann book [105] (p. 240): "Erwähnt sei noch, daß das modifizierte Newtonsche und das elastische Anziehungsgesetz die einzigen Zentralkräfte sind, die im sphärischen Raum immer, im hyperbolischen, sobald die Konstante des Flächensaztes in gewissen Grenzen bleibt, auf geschlossene Bahnkurven führen. Die Methode, mit deren Hilfe Bertrand den entsprechenden Satz der euklidischen Geometrie bewiesen hat, ist auf die nichteuklidische Geometrie leicht übertragbar", but there is no a corresponding proof. This proof appeared in the second edition (1912) of the Liebmann book for the hyperbolic space and in the third edition (1923) the paragraph concerning the spherical case was added.

In [127] a one-particle motion under the action of a central force in spaces of constant curvature was considered from a point of view, different from the common concept of natural mechanical systems on Riemannian spaces, accepted particularly in the present book (see Sect. 4.3.3). As a consequence there are no explicit formulas for a particle trajectory even for the potential V_c. Also, there are no nontrivial potentials for which all bounded trajectories are closed.

One can consider the classical mechanics in constant curvature spaces as a predecessor of special and general relativity. After the rise of these theories the above-mentioned papers of Schering, Killing and Liebmann were almost completely forgotten. Note that the description of a particle motion in central potentials in spaces $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ was shorten in the second and the third editions of the Liebmann book w.r.t. the first edition in favor of special relativity.

Similar models attracted attention later from the point of view of quantum mechanics and the theory of integrable dynamical systems.

Quantum mechanical spectral problem on the sphere $\mathbf{S}^3$ for potential V_c (Coulomb problem) was solved by Schrödinger in 1940 by the factorization (ladder) method, invented by himself [156]. Stevenson in 1941 solved the same problem using more traditional analysis of the hypergeometric differential equation [185] (see also the Infeld result in 1941 [72]). Infeld and Schild in 1945 solved a similar problem in the space $\mathbf{H}^3(\mathbb{R})$ [73] (see also [74]). Note that Schrödinger, Stevenson, Infeld and Schild did not cite the works of Schering, Killing, Liebmann and probably did not know them.

Nishino in 1972 [130] (see also [70]) found all central potential in constant curvature spaces corresponding to classical one-body problems admitting an additional integral, quadric in momenta (analogs of the Runge-Lenz vector), which is independent from integrals linear in momenta. These potentials again appeared to be V_c and V_o. He pointed out that the corresponding one-particle systems are analogs for the isotropic harmonic oscillator and the Kepler problem in Euclidean case and calculated the Poisson brackets for integrals of motion. However, Nishino did not notice the connection of the considered problem with the Bertrand one in Euclidean space and did not cite any of his predecessors mentioned above.

Further the period of partial rediscovery started. The generalized Bertrand problem in the space $\mathbf{S}^n$ was resolved in 1979 by Higgs[8] [69]. In the same paper there were found one-particle energy levels for the potential V_o on $\mathbf{S}^2$ and additional integrals both in classical and quantum cases. The latter integrals are generalization of the Runge-Lenz vector from Euclidean case. In the same year they were independently obtained by Kurochkin and Otchik for the sphere $\mathbf{S}^3$ [98]. In 1980 Bogush, Kurochkin and Otchik found them also for the hyperbolic space $\mathbf{H}^3(\mathbb{R})$ [21]. The same problem for the spaces $\mathbf{S}^n$, $n \geqslant 3$ and potentials V_c, V_o was solved by Leemon in 1980 [101]. Coordinate systems admitting variable separation for the quantum Kepler problem in the spaces $\mathbf{S}^3, \mathbf{H}^3(\mathbb{R})$ were found by Bogush, Otchik and Red'kov in 1983 [20] (see also [139]). Among their predecessors these authors cited only Schrödinger, Infeld and Schild.

The generalized Bertrand problem in the space $\mathbf{S}^3$ was solved once again in 1980 by Slawianowski [177]. In the same year Slawianowski and Slominski carried out the quasiclassical quantization of the one-particle motion in $\mathbf{S}^3$ for potentials V_c and V_o, without using the Maslov index [179]. These authors did not cite papers of their predecessors.

In 1982 Ikeda and Katayama solved once again the generalized Bertrand problem in the spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$ in a uniform way [71], cited only [130]. In the paper [84] the application of these potentials to some cosmological models was studied. In [81] (1990) Katayama showed the solvability of the one-particle spectral problem for the potential V_c on the space $\mathbf{H}^3(\mathbb{R})$ by the ladder method. This paper was the first one on this subject, where such a big

[8] See the remark above on the Stäckel paper [182] and the Liebmann book [105].

number of previous papers were cited: [21, 69, 72, 98, 101, 130, 156, 185]. In the paper [82] (1992) Katayama carried out the similar study for the oscillator potential V_o on constant curvature spaces.

The connection of the Runge-Lenz operator for the quantum Kepler problem in $\mathbf{S}^3$ with the Schrödinger ladder method was discussed by Barut and Wilson in 1985 [14]. In 1987 and 1990 Barut, Inomata and Junker solved the Kepler problem in $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ using the functional integration [12].

In 1991 Dombrowski and Zitterbarth published the survey [42], where described almost forgotten results of Schering, Killing and Liebmann. On the other hand, it seems that Dombrowski and Zitterbarth were not aware of Schrödinger, Infeld, Nishino, Higgs, Leemon and later papers. In particular, Zitterbarth in his Ph.D. thesis rediscovered the Runge-Lenz vector for potential V_c in constant curvature spaces.

In papers [137, 138] (1991 and 1994) Otchik considered the one-particle quantum two-center Coulomb problem in $\mathbf{S}^3$ and found a coordinate system admitting the variable separation. The corresponding ordinary differential equations are reduced to the Heun equation. He did not cite the Killing paper [87], containing similar results for the classical case.

In 1992 Granovskii, Zhedanov and Lutsenko developed the algebraic approach of [21, 69, 98, 101] to one-particle problems for potentials V_c and V_o in the spaces $\mathbf{S}^n, \mathbf{H}^n(\mathbb{R})$ [54].

Potentials V_c and V_o as solutions of the generalized Bertrand problem in $\mathbf{S}^n$ were rediscovered one more time in 1992 by Kozlov and Harin [95]. The two-body classical mechanical problem with the potential V_c seems to be mentioned here for the first time. There was declared a conjecture that not all orbits of this problem are closed. Also, Kozlov and Harin rediscovered there the Killing result [87] on the integrability of the classical two-center Kepler problem on the sphere $\mathbf{S}^2$. Note that this integrability is also a direct consequence of the Otchik result (1991) cited above on the separation of variables for the corresponding quantum problem. In 1994 Kozlov rediscovered also the Kepler laws in spaces of constant curvature [93]. Among his predecessors only Slawianowski and Slominski were pointed out. In the paper [94] of the same year Kozlov and Fedorov proved the integrability of the classical one-particle motion on the sphere $\mathbf{S}^n$ in a superposition of $2(n+1)$ potentials V_o with their centers at points

$$(\pm 1, 0, \ldots, 0),\ (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)$$

for the standard $\mathbf{S}^n$ model in the space $\mathbb{R}^{n+1}$ defined by the equation

$$\sum_{i=0}^{n} x_i^2 = 1 \ .$$

In 1992 Chernikov considered the potential V_c in the space $\mathbf{H}^3(\mathbb{R})$ as a spherically symmetric harmonic function and the corresponding one-body motion, i.e., partially rediscovered again the results of Killing and Liebmann.

A variable separation for a one-particle Schrödinger operator and some noncentral potentials in the spaces $\mathbf{S}^2$ and $\mathbf{H}^2(\mathbb{R})$ was studied by Kalnins, Miller, Hakobyan and Pogosyan in 1996–1998 [78, 79, 80].

In 1996 the first paper of the author concerning mechanics on constant curvature spaces was published [159]. At that moment he knew only papers [93, 95, 177, 179] on this subject. The new results in [159] w.r.t. preceding papers were the semiclassical quantization of a one-body motion in potentials V_c and V_o for the space $\mathbf{H}^3(\mathbb{R})$ using the Maslov index and the proof of the self-adjointness of the corresponding Schrödinger operator. Note that the problem of self-adjointness of these operator was not considered before, therefore calculations of energy levels were not completely satisfactory from the mathematical point of view.

The two-body problem with a central interaction on constant curvature spaces was considered for the first time by the author in 1998 [160]. Its Hamiltonian reduction to the system with two degrees of freedom was carried out by explicit coordinate calculations. For some potentials there was proved the solvability of the reduced problem for an infinite period of time. The author's paper [166], dealing with the same questions for the spaces $\mathbf{S}^2$ and $\mathbf{H}^2(\mathbb{R})$, was submitted for publication in 1997, but printed only in 2000. In 1999 the author considered the quantum mechanical two-body problem in the spaces $\mathbf{S}^2$ and $\mathbf{H}^2(\mathbb{R})$, studied the self-adjointness of the corresponding Schrödinger operator and found in explicit form some its infinite energy level series for $\mathbf{S}^2$, corresponding to some central potentials [162].

In 1999 Kilin considered some partial solutions of the two-body problem with the potential V_c in the spaces $\mathbf{S}^2$ and $\mathbf{H}^2(\mathbb{R})$, corresponding to the motion of both bodies along circles with a common center in such a way that bodies are diametrically opposite w.r.t. this center. By explicit calculations he proved the stability of these solutions in linear approximation that does not guarantee the real stability. Besides, there were found points of the relative equilibrium of a "light" third body in the potential of two "heavy" ones, rotating in the way, described above. In the same year Chernoivan and Mamaev considered the restricted classical two-body problem in the spaces $\mathbf{S}^2$ and $\mathbf{H}^2(\mathbb{R})$ for the potential V_c, i.e., the problem of a "light" body motion about a "heavy" one, which moves along a geodesic with a constant velocity. Numerical calculations of the Poincaré surfaces of section for this problem demonstrated its nonintegrability. Among the preceding papers on this subject the authors of [38, 86] mentioned only [93] and [95]. These results were included in the book [27].

In 2000–2002 Vozmischeva and Oshemkov continued studying integrable two-center Kepler problem on the sphere $\mathbf{S}^2$ [203, 205], not knowing Killing ([87]) and Otchik ([137, 138]) results.

Results of Schrödinger [156], Infeld [72], Slawianowski and Slominski [177, 179], Kozlov, Harin [93, 95], Shchepetilov [160] was considered in the Slawianowski review [178] in 2000. Some basic results of Killing and Liebmann for classical mechanics on the spaces $\mathbf{S}^3$ and $\mathbf{H}^3(\mathbb{R})$ are contained in [198].

In 2000 the author derived an explicitly invariant expression for the two-body quantum mechanical Hamiltonian with central interaction in the spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$ via a radial differential operator and some invariant operators on the spaces $\mathbf{S}^n_S$ and $\mathbf{H}^n(\mathbb{R})_S$ [163]. In the same paper the invariant Hamiltonian reduction of the two-body classical problem in the spaces

$\mathbf{S}^n$, $\mathbf{H}^n(\mathbb{R})$, $n = 2, 3$ was carried out. In [184] the author and Stepanova studied the self-adjointness of the corresponding Schrödinger operators and found some energy level series in an explicit form for $\mathbf{S}^3$, corresponding to some central potentials. In 2003 these results were partially generalized by the author for two-point homogeneous spaces [168, 169]. These results after correction of some errors are included in the present monograph.

In 2001 and 2003 Ziglin proved in [213, 214] the meromorphic nonintegrability of the restricted two-body problem on the sphere $\mathbf{S}^2$ with the potentials V_c and V_o in the class of meromorphic functions. Similar results with smaller restrictions, valid also for the restricted two-body problem on the hyperbolic plane $\mathbf{H}^2(\mathbb{R})$, were obtained a little later by Maciejewski and Przybylska in [112]. In 2006 ([171]) the author proved the meromorphic nonintegrability of the nonrestricted reduced two-body problem on the spaces $\mathbf{S}^2$ and $\mathbf{H}^2(\mathbb{R})$ with potentials V_c and V_o.

In 2003 Bogush, Kurochkin and Otchik considered in [22] the Coulomb scattering in the space $\mathbf{H}^3(\mathbb{R})$.

7

Classical Two-Body Problem on Two-Point Homogeneous Riemannian Spaces

In this chapter we consider the classical two-body problem on two-point homogeneous Riemannian spaces. First we derive an expression of its Hamiltonian function through the canonical variables r, p_r, corresponding to the radial degree of freedom, and generators of the Poisson algebra $\mathcal{P}_I(Q_{\mathbf{S}}) \cong \operatorname{gr}\operatorname{Diff}_I(Q_{\mathbf{S}})$. Here we use the dequantization procedure from Sect. 4.3.3 since it allows one to avoid quite cumbersome calculations similar to those in Chap. 5. Also, one gets integrals of motion from central elements of the algebra $\operatorname{Diff}_I(Q_{\mathbf{S}})$, constructed in Chap. 3.

Then in Sect. 7.3 we derive the conditions for the existence of global solutions for the two-body problem and discuss the problem of its integrability. Section 7.4 contains the consideration of the existing center of mass concepts on constant curvature spaces and their connections with constructed expressions of two-body Hamiltonian functions. In Sect. 7.5 we study the Hamiltonian reduction of the two-body problem, restricting ourselves with constant curvature spaces and classify reduced Hamiltonian systems.

7.1 Explicitly Invariant Form of the Hamiltonian Two-Body Function for Compact Two-Point Homogeneous Spaces

Consider the classical mechanical two-body problem on an arbitrary two-point homogeneous space Q. The configuration space for this problem is $(Q \times Q)\setminus \operatorname{diag}$, since a collision of pointlike particles leads to an uncertainty of their motion.[1] Here we will use notations from Sect. 5.1.

The submanifold $Q_{op} \subset Q \times Q$ was defined there as the set of pairs $(\mathbf{x}, \mathbf{y}) \in Q \times Q$ such that $\rho(\mathbf{x}, \mathbf{y}) = \operatorname{diam} Q$. In Sect. 5.1 the space $(Q \times Q)\setminus(\operatorname{diag} \cup Q_{op})$ was represented as the direct product $I \times (G/K_0)$, where $I = (0, \operatorname{diam} Q) \subset \mathbb{R}$. Similarly, for noncompact two-point homogeneous space Q one has $(Q \times Q)\setminus \operatorname{diag} = I \times (G/K_0)$, where $I = (0, +\infty)$. This means that the phase

[1] The excluding of the diagonal from $Q \times Q$ does not solve the collision problem, since for some initial conditions global solutions can be absent.

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space of the classical two-body problem can be represented as

$$T^*I \times T^*(G/K_0) \cong T^*I \times T^*Q_S \tag{7.1}$$

in the noncompact case and as

$$(T^*I \times T^*Q_S) \cup \widetilde{T}^*Q_{op} =: M_{ess} \cup \widetilde{T}^*Q_{op} \tag{7.2}$$

in the compact case, where $\widetilde{T}^*Q_{op}$ is the restriction of the cotangent bundle $T^*(Q \times Q)$ onto Q_{op}.

Since $\dim Q_{op} = \dim Q + \dim A_x$, it can be easily verified that $\dim T^*((Q \times Q)\setminus \mathrm{diag}) - \dim \widetilde{T}^*Q_{op} > 1$, if $\dim Q - \dim A_x > 1$. The last inequality is valid for $Q \neq \mathbf{P}^n(\mathbb{R})$ (see Sect. 1.2); in this case the subspace $\widetilde{T}^*Q_{op}$ does not separate $T^*((Q \times Q)\setminus \mathrm{diag})$ and most trajectories of the two-body problem do not intersect the subspace $\widetilde{T}^*Q_{op}$. Therefore, many properties of the two-body problem on a compact two-point homogeneous space Q (for instance, the integrability and the collision problem) can be studied after the restriction of this system onto the space $T^*I \times T^*Q_S$, at least for $Q \neq \mathbf{P}^n(\mathbb{R})$.

First of all, using the dequantization procedure described in Sect. 4.3.3 and expressions for the two-body quantum mechanical Hamiltonians on two-point homogeneous spaces, one can derive corresponding expressions for two-body Hamiltonian functions on the space $T^*I \times T^*(G/K_0)$. Let p_r be a momentum on T^*I, corresponding to the coordinate r on I.

7.1.1 Quaternionic Case

From expression (5.22) one gets for $Q = \mathbf{P}^n(\mathbb{H})$ the two-body Hamiltonian function in the form:

$$\begin{aligned} h = {} & \frac{(1+r^2)^2}{8mR^2} p_r^2 + \frac{(m_1\alpha - m_2\beta)(1+r^2)}{2m_1m_2R^2} p_r p_0 + \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2} p_0^2 \\ & + \frac{1}{2}\left(D_s p_1 + F_s p_2 + 2E_s p_3 + C_s p_4 + A_s p_5 + 2B_s p_6\right) + V(r), \end{aligned} \tag{7.3}$$

where p_i are generators of the Poisson algebra $\mathcal{P}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right) \cong \mathrm{gr}\,\mathrm{Diff}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right)$, corresponding to generators D_i, $i = 0, \ldots, 10$ of the algebra $\mathrm{Diff}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right)$, and functions $A_s, B_s, C_s, D_s, F_s, E_s$ were defined in Sect. 5.2.

Due to the definition of the Poisson structure for a graded algebra $\mathrm{gr}\,U(\mathfrak{g})$ in Sect. 4.2.1 and commutative relations (3.15) one gets 55 commutative relations for generators of the Poisson algebra $\mathcal{P}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right)$ (see Remark 4.3):

$$\begin{aligned} & [p_0, p_1]_P = -p_3,\ [p_0, p_2]_P = p_3,\ [p_0, p_3]_P = \frac{1}{2}(p_1 - p_2), \\ & [p_0, p_4]_P = -2p_6,\ [p_0, p_5]_P = 2p_6,\ [p_0, p_6]_P = p_4 - p_5, \\ & [p_0, p_7]_P = -p_8,\ [p_0, p_8]_P = p_7,\ [p_0, p_9]_P = 0,\ [p_0, p_{10}]_P = 0, \\ & [p_1, p_2]_P = -2p_0p_3 - 2p_7,\ [p_1, p_3]_P = -p_0p_1 + p_8,\ [p_1, p_4]_P = 2p_7, \\ & [p_1, p_5]_P = 0,\ [p_1, p_6]_P = p_8,\ [p_1, p_7]_P = -p_3p_6 - p_1p_4 + p_9 + p_{10}, \end{aligned}$$

$$
\begin{aligned}
&[p_1,p_8]_P = -p_3p_5 - p_1p_6,\ [p_1,p_9]_P = -2p_3p_8 - 2p_1p_7,\\
&[p_1,p_{10}]_P = p_6p_8 - p_5p_7,\ [p_2,p_3]_P = p_0p_2 + p_8,\\
&[p_2,p_4]_P = -2p_7,\ [p_2,p_5]_P = 0,\ [p_2,p_6]_P = -p_8,\\
&[p_2,p_7]_P = -p_3p_6 + p_2p_4 - p_9 - p_{10},\ [p_2,p_8]_P = -p_3p_5 + p_2p_6,\\
&[p_2,p_9]_P = -2p_3p_8 + 2p_2p_7,\ [p_2,p_{10}]_P = -p_6p_8 + p_5p_7,\\
&[p_3,p_4]_P = 0,\ [p_3,p_5]_P = 2p_8,\ [p_3,p_6]_P = p_7,\ [p_3,p_7]_P = -\frac{1}{2}(p_1+p_2)p_6,
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
&[p_3,p_8]_P = -\frac{1}{2}(p_1+p_2)p_5 + p_9 + p_{10},\ [p_3,p_9]_P = -(p_1+p_2)p_8,\\
&[p_3,p_{10}]_P = p_6p_7 - p_4p_8,\ [p_4,p_5]_P = -4p_0p_6,\ [p_4,p_6]_P = -2p_0p_4,\\
&[p_4,p_7]_P = (p_1-p_2)p_4,\ [p_4,p_8]_P = (p_1-p_2)p_6 - 2p_0p_7,\\
&[p_4,p_9]_P = 2(p_1-p_2)p_7,\ [p_4,p_{10}]_P = 0,\ [p_5,p_6]_P = 2p_0p_5,\\
&[p_5,p_7]_P = 2p_3p_6 + 2p_0p_8,\ [p_5,p_8]_P = 2p_3p_5,\ [p_5,p_9]_P = 4p_3p_8,\ [p_5,p_{10}]_P = 0,\\
&[p_6,p_7]_P = \frac{1}{2}(p_1-p_2)p_6 + p_3p_4 + p_0p_7,\\
&[p_6,p_8]_P = \frac{1}{2}(p_1-p_2)p_5 + p_3p_6 - p_0p_8,\\
&[p_6,p_9]_P = (p_1-p_2)p_8 + 2p_3p_7,\ [p_6,p_{10}]_P = 0,\\
&[p_7,p_8]_P = \frac{1}{2}(p_1-p_2)p_8 - p_3p_7 - p_0(p_9+p_{10}),\\
&[p_7,p_9]_P = (p_1-p_2)(p_9+p_{10}),\ [p_7,p_{10}]_P = \frac{1}{2}(p_2-p_1)p_6^2\\
&\qquad - p_0p_6p_7 + p_0p_4p_8 + \frac{1}{2}(p_1-p_2)p_4p_5,\ [p_8,p_9]_P = 2p_3(p_9+p_{10}),\\
&[p_8,p_{10}]_P = -p_3p_6^2 + p_0p_6p_8 - p_0p_5p_7 + p_3p_4p_5,\\
&[p_9,p_{10}]_P = (p_5p_7 - p_6p_8)(p_1-p_2) + 2p_3p_4p_8 - 2p_3p_6p_7\ .
\end{aligned}
$$

The relation (3.13) is transformed into

$$p_{10}^2 - p_4p_5p_9 - 2p_6p_7p_8 + p_9p_6^2 + p_4p_8^2 + p_5p_7^2 = 0\ .$$

Likewise, the additional relation (3.14) in the case $n = 2$ now becomes

$$p_1p_2 - p_3^2 - p_9 = 0\ , \tag{7.5}$$

and can be used in this case for excluding p_9 from the list of generators.

It is easily seen that all Poisson algebras $\mathcal{P}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right)$ for $n \geqslant 3$ are isomorphic to each other since the dependence on n was eliminated in commutative relations by dequantization procedure:

$$\mathrm{Diff}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right) \to \mathcal{P}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right)\ .$$

The same is also valid for Hamiltonian functions.

Elements C_i^{gr} from $\mathrm{Z}\,\mathcal{P}_I\left(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}}\right)$, $i=1,2,3$, corresponding to elements C_i, $i=1,2,3$ from Sect. 3.2.2, can be obtained by the rejection of terms with degrees lower than $\deg C_i$ in the expression for C_i. This implies the following expressions

$$
\begin{aligned}
C_1^{\mathrm{gr}} &= p_0^2+p_1+p_2+p_4+p_5,\ C_2^{\mathrm{gr}}=p_1p_2-p_3^2-p_9,\ C_3^{\mathrm{gr}}=\frac{1}{2}(p_1+p_2)(p_4+p_5)\\
&+\frac{1}{4}(p_1-p_2)^2+p_3^2+\frac{1}{4}(p_4-p_5)^2+p_6^2+p_9-2p_{10}\\
&+\frac{1}{2}p_0^2(p_1+p_2+p_4+p_5)+\frac{1}{4}p_0^4\,.
\end{aligned}
$$

Note that due to (7.5) it holds $C_2^{\mathrm{gr}}=0$ for $n=2$.

7.1.2 Octonionic Case

The expression for the two-body Hamiltonian function for $Q=\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}$ is again (7.3), where p_i, $i=0,\dots,9$ are now generators of the Poisson algebra

$$\mathcal{P}_I\left(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}\right)\cong \mathrm{gr}\,\mathrm{Diff}_I\left(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}\right),$$

corresponding to generators D_i, $i=0,\dots,9$ of the algebra $\mathrm{Diff}_I\left(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}\right)$.
Commutative relations for them have the form

$$
\begin{aligned}
&[p_0,p_1]_P=-p_3,\ [p_0,p_2]_P=p_3,\ [p_0,p_3]_P=\frac{1}{2}(p_1-p_2),\ [p_0,p_4]_P=-2p_6,\\
&[p_0,p_5]_P=2p_6, [p_0,p_6]_P=p_4-p_5,\ [p_0,p_7]_P=-p_8, [p_0,p_8]_P=p_7,\\
&[p_0,p_9]_P=0,\ [p_1,p_2]_P=-2p_0p_3-2p_7,\ [p_1,p_3]_P=-p_0p_1+p_8,\\
&[p_1,p_4]_P=2p_7,\ [p_1,p_5]_P=0,\ [p_1,p_6]_P=p_8, [p_1,p_7]_P=p_1(p_2-p_4)-p_9\\
&\qquad -p_3p_6-p_3^2,\ [p_1,p_8]_P=-p_3p_5-p_1p_6,\ [p_1,p_9]_P=p_5p_7-p_6p_8,\\
&[p_2,p_3]_P=p_0p_2+p_8,\ [p_2,p_4]_P=-2p_7,\ [p_2,p_5]_P=0,\ [p_2,p_6]_P=-p_8,\\
&[p_2,p_7]_P=(p_4-p_1)p_2+p_9-p_3p_6+p_3^2,\ [p_2,p_8]_P=p_2p_6-p_3p_5,\\
&[p_2,p_9]_P=p_6p_8-p_5p_7,\ [p_3,p_4]_P=0,\ [p_3,p_5]_P=2p_8,\\
&[p_3,p_6]_P=p_7,\ [p_3,p_7]_P=-\frac{1}{2}(p_1+p_2)p_6,\\
&[p_3,p_8]_P=p_1p_2-\frac{1}{2}(p_1+p_2)p_5-p_9-p_3^2,\ [p_3,p_9]_P=p_4p_8-p_6p_7, \qquad (7.6)\\
&[p_4,p_5]_P=-4p_0p_6,\ [p_4,p_6]_P=-2p_0p_4,\\
&[p_4,p_7]_P=(p_1-p_2)p_4,\ [p_4,p_8]_P=(p_1-p_2)p_6-2p_0p_7,\\
&[p_4,p_9]_P=0,\ [p_5,p_6]_P=2p_0p_5,\ [p_5,p_7]_P=2p_3p_6+2p_0p_8,\\
&[p_5,p_8]_P=2p_3p_5,\ [p_5,p_9]_P=0,\ [p_6,p_7]_P=\frac{1}{2}(p_1-p_2)p_6+p_3p_4+p_0p_7,\\
&[p_6,p_8]_P=\frac{1}{2}(p_1-p_2)p_5+p_3p_6-p_0p_8,\ [p_6,p_9]_P=0,\\
&[p_7,p_8]_P=\frac{1}{2}(p_1-p_2)p_8-p_3p_7+p_0p_9+p_0p_3^2-p_0p_1p_2,
\end{aligned}
$$

$$[p_7, p_9]_P = \frac{1}{2}(p_2 - p_1)p_4p_5 + p_0p_6p_7 - p_0p_4p_8 + \frac{1}{2}(p_1 - p_2)p_6^2 \,,$$
$$[p_8, p_9]_P = p_3p_6^2 - p_0p_6p_8 - p_3p_4p_5 + p_0p_5p_7 \,.$$

Elements C_i^{gr}, $i = 1, 2$ from $\mathrm{Z}\,\mathcal{P}_I\left(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}\right)$, corresponding to elements C_1, C_2 from Sect. 3.5.2, are

$$C_1^{\mathrm{gr}} = p_0^2 + p_1 + p_2 + p_4 + p_5,\; C_2^{\mathrm{gr}} = p_4p_5 - p_6^2 - 2p_9 \,.$$

7.1.3 Complex Case

From expression (5.23) one gets for $Q = \mathbf{P}^n(\mathbb{C})$ the two-body Hamiltonian function

$$\begin{aligned} h = {} & \frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1+r^2)}{2m_1m_2R^2}p_rp_0 + \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}p_0^2 \\ & + \frac{1}{2}\left(D_sp_1 + F_sp_2 + 2E_sp_3 + C_sp_4^2 + A_sp_5^2 + 2B_sp_4p_5\right) + V(r), \end{aligned} \tag{7.7}$$

where p_i, $i = 0, \ldots, 5$ are generators of the Poisson algebra $\mathcal{P}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}}) \cong \mathrm{gr}\,\mathrm{Diff}_I(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$, corresponding to generators D_i, $i = 0, \ldots, 5$ of the algebra $\mathrm{Diff}_I\,(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$. Denote the last generators of $\mathcal{P}_I\,(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$, corresponding to the generator $\Box$ of $\mathrm{Diff}_I\,(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$, by $p_\Box$.

The commutative relations for these generators of $\mathcal{P}_I\,(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ are as follows

$$\begin{aligned} & [p_0, p_1]_P = -p_3,\; [p_0, p_2]_P = p_3,\; [p_0, p_3]_P = \frac{1}{2}(p_1 - p_2), \\ & [p_0, p_4]_P = -p_5,\; [p_0, p_5]_P = p_4, [p_0, p_\Box]_P = 0,\; [p_1, p_2]_P = -2p_0p_3 - 2p_\Box p_4, \\ & [p_1, p_3]_P = -p_0p_1 + p_\Box p_5,\; [p_1, p_4]_P = p_\Box, [p_1, p_5]_P = 0, \\ & [p_1, p_\Box]_P = -p_1p_4 - p_3p_5,\; [p_2, p_3]_P = p_0p_2 + p_\Box p_5,\; [p_2, p_4]_P = -p_\Box, \\ & [p_2, p_5]_P = 0,\; [p_2, p_\Box]_P = p_2p_4 - p_3p_5,\; [p_3, p_4]_P = 0,\; [p_3, p_5]_P = p_\Box, \\ & [p_3, p_\Box]_P = -\frac{1}{2}(p_1 + p_2)p_5,\; [p_4, p_5]_P = -p_0, \\ & [p_4, p_\Box]_P = \frac{1}{2}(p_1 - p_2),\; [p_5, p_\Box]_P = p_3 \,. \end{aligned} \tag{7.8}$$

The additional relation (3.28) in the case $n = 2$ now becomes

$$p_1p_2 - p_3^2 - p_\Box^2 = 0 \,. \tag{7.9}$$

Elements $C_i^{\mathrm{gr}} \in \mathrm{Z}\,\mathcal{P}_I\,(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})\,$, $i = 1, 2, 3$, corresponding to elements C_i, $i = 1, 2, 3$ from Sect. 3.3.2, are

$$\begin{aligned} & C_1^{\mathrm{gr}} = p_0^2 + p_1 + p_2 + p_4^2 + p_5^2,\; C_2^{\mathrm{gr}} = (p_1 - p_2)p_5 - 2p_3p_4 + 2p_0p_\Box, \\ & C_3^{\mathrm{gr}} = p_1p_2 - p_3^2 - p_\Box^2 \,. \end{aligned}$$

Note that due to (7.9) it holds $C_3^{\mathrm{gr}} = 0$ for $n = 2$.

7.1.4 Real Case

From expression (5.24) one gets for $Q = \mathbf{P}^n(\mathbb{R}), \mathbf{S}^n, n \geqslant 3$ the two-body Hamiltonian function:

$$\begin{aligned} h = &\frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1+r^2)}{2m_1m_2R^2}p_r p_0 + \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}p_0^2 \\ &+ \frac{1}{2}\left(C_s p_1 + A_s p_2 + 2B_s p_3\right) + V(r) \, . \end{aligned} \tag{7.10}$$

The Poisson algebra $\mathcal{P}_I\left(\mathbf{P}^n(\mathbb{R})_{\mathbf{S}}\right) \cong \mathcal{P}_I\left(\mathbf{S}^n_{\mathbf{S}}\right)$ is generated by elements p_i, $i = 0, \ldots, 3$ if $n \geqslant 4$ and by elements $p_\Box, p_i$, $i = 0, \ldots, 3$ if $n = 3$. The commutative relations for these generators are

$$\begin{aligned} &[p_0, p_1]_P = -2p_3, \ [p_0, p_2]_P = 2p_3, \ [p_0, p_3]_P = p_1 - p_2, \\ &[p_1, p_2]_P = -4p_0p_3, \ [p_1, p_3]_P = -2p_0p_1, \ [p_2, p_3]_P = 2p_0p_2 \, . \end{aligned} \tag{7.11}$$

The generator $p_\Box$, if exists, commutes with all other generators. The additional relation (3.36) in the case $n = 3$ now looks like

$$p_1p_2 - p_3^2 - p_\Box^2 = 0 \, . \tag{7.12}$$

Elements $C_i^{\mathrm{gr}} \in \mathrm{Z}\,\mathcal{P}_I\left(\mathbf{P}^n(\mathbb{R})_{\mathbf{S}}\right)$, $i = 1, 2$, corresponding to elements C_i, $i = 1, 2$ from Sect. 3.4.2, are

$$C_1^{\mathrm{gr}} = p_0^2 + p_1 + p_2, \ C_2^{\mathrm{gr}} = p_1p_2 - p_3^2 \, .$$

Note that due to (7.12) it holds $C_2^{\mathrm{gr}} = p_\Box^2$ for $n = 3$.

The expression (5.25) implies for $Q = \mathbf{P}^2(\mathbb{R}), \mathbf{S}^2$ that

$$\begin{aligned} h = &\frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1+r^2)}{2m_1m_2R^2}p_r p_0 + \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}p_0^2 \\ &+ \frac{1}{2}\left(C_s p_1^2 + A_s p_2^2 + 2B_s p_1p_2\right) + V(r), \end{aligned} \tag{7.13}$$

where

$$[p_0, p_1]_P = -p_2, \ [p_0, p_2]_P = p_1, \ [p_1, p_2]_P = -p_0 \, . \tag{7.14}$$

The Poisson algebra $\mathcal{P}_I\left(\mathbf{P}^2(\mathbb{R})_{\mathbf{S}}\right) \cong \mathcal{P}_I\left(\mathbf{S}^2_{\mathbf{S}}\right)$ is isomorphic to the algebra $\operatorname{gr} U(\mathfrak{so}(3))$. In this simplest case there is only one central independent element from $\operatorname{gr} U(\mathfrak{so}(3))$: $C_1^{\mathrm{gr}} = p_0^2 + p_1^2 + p_2^2$.

7.2 Explicitly Invariant Form of the Hamiltonian Two-Body Function for Noncompact Two-Point Homogeneous Spaces

Here we shall derive expressions of the Hamiltonian function for the two-body system on a noncompact two-point homogeneous spaces Q through generators of the corresponding Poisson algebra $\mathcal{P}_I(Q_S)$. Also, there will be obtained relations for these generators. Considerations here are completely similar to considerations in the previous section.

7.2.1 Quaternionic Case

From expression (5.26) one gets for $Q = \mathbf{P}^n(\mathbb{H})$ the two-body Hamiltonian function in the form:

$$\begin{aligned} h = &\frac{(1-r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1-r^2)}{2m_1m_2R^2}p_r\bar{p}_0 + \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}\bar{p}_0^2 \\ &+ \frac{1}{2}\left(D_h\bar{p}_1 + F_h\bar{p}_2 + 2E_h\bar{p}_3 + C_h\bar{p}_4 + A_h\bar{p}_5 + 2B_h\bar{p}_6\right) + V(r) \ , \end{aligned} \tag{7.15}$$

where $\bar{p}_i$ are generators of the Poisson algebra $\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{H})_\mathbf{S}\right) \cong \operatorname{gr}\operatorname{Diff}_I\left(\mathbf{H}^n(\mathbb{H})_\mathbf{S}\right)$, corresponding to generators $\bar{D}_i$, $i = 0,\dots,10$ of the algebra $\operatorname{Diff}_I\left(\mathbf{P}^n(\mathbb{H})_\mathbf{S}\right)$, and functions $A_h, B_h, C_h, D_h, F_h, E_h$ were defined in Sect. 5.3.

From commutative relations (3.17) one gets 55 commutative relations for generators of the Poisson algebra $\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{H})_\mathbf{S}\right)$ (see Remark 4.3):

$$\begin{aligned}
&[\bar{p}_0,\bar{p}_1]_P = \bar{p}_3,\ [\bar{p}_0,\bar{p}_2]_P = \bar{p}_3,\ [\bar{p}_0,\bar{p}_3]_P = \frac{1}{2}(\bar{p}_1+\bar{p}_2),\ [\bar{p}_0,\bar{p}_4]_P = 2\bar{p}_6,\\
&[\bar{p}_0,\bar{p}_5]_P = 2\bar{p}_6, [\bar{p}_0,\bar{p}_6]_P = \bar{p}_4+\bar{p}_5, [\bar{p}_0,\bar{p}_7]_P = \bar{p}_8, [\bar{p}_0,\bar{p}_8]_P = \bar{p}_7,\\
&[\bar{p}_0,\bar{p}_9]_P = 0,\ [\bar{p}_0,\bar{p}_{10}]_P = 0, [\bar{p}_1,\bar{p}_2]_P = -2\bar{p}_0\bar{p}_3 - 2\bar{p}_7,\\
&[\bar{p}_1,\bar{p}_3]_P = -\bar{p}_0\bar{p}_1 - \bar{p}_8,\ [\bar{p}_1,\bar{p}_4]_P = -2\bar{p}_7,\ [\bar{p}_1,\bar{p}_5]_P = 0,\\
&[\bar{p}_1,\bar{p}_6]_P = -\bar{p}_8,\ [\bar{p}_1,\bar{p}_7]_P = \bar{p}_3\bar{p}_6 - \bar{p}_1\bar{p}_4 - \bar{p}_9 - \bar{p}_{10},\ [\bar{p}_1,\bar{p}_8]_P = \bar{p}_3\bar{p}_5 - \bar{p}_1\bar{p}_6,\\
&[\bar{p}_1,\bar{p}_9]_P = 2\bar{p}_3\bar{p}_8 - 2\bar{p}_1\bar{p}_7,\ [\bar{p}_1,\bar{p}_{10}]_P = \bar{p}_5\bar{p}_7 - \bar{p}_6\bar{p}_8,\ [\bar{p}_2,\bar{p}_3]_P = \bar{p}_0\bar{p}_2 + \bar{p}_8,\\
&[\bar{p}_2,\bar{p}_4]_P = -2\bar{p}_7,\ [\bar{p}_2,\bar{p}_5]_P = 0,\ [\bar{p}_2,\bar{p}_6]_P = -\bar{p}_8,\\
&[\bar{p}_2,\bar{p}_7]_P = \bar{p}_2\bar{p}_4 - \bar{p}_3\bar{p}_6 - \bar{p}_9 - \bar{p}_{10},\ [\bar{p}_2,\bar{p}_8]_P = \bar{p}_2\bar{p}_6 - \bar{p}_3\bar{p}_5,\\
&[\bar{p}_2,\bar{p}_9]_P = 2\bar{p}_2\bar{p}_7 - 2\bar{p}_3\bar{p}_8,\ [\bar{p}_2,\bar{p}_{10}]_P = \bar{p}_5\bar{p}_7 - \bar{p}_6\bar{p}_8,\\
&[\bar{p}_3,\bar{p}_4]_P = 0,\ [\bar{p}_3,\bar{p}_5]_P = 2\bar{p}_8,\ [\bar{p}_3,\bar{p}_6]_P = \bar{p}_7,\ [\bar{p}_3,\bar{p}_7]_P = \frac{1}{2}(\bar{p}_2-\bar{p}_1)\bar{p}_6,
\end{aligned} \tag{7.16}$$

$$\begin{aligned}
&[\bar{p}_3,\bar{p}_8]_P = \frac{1}{2}(\bar{p}_2-\bar{p}_1)\bar{p}_5 + \bar{p}_9 + \bar{p}_{10},\ [\bar{p}_3,\bar{p}_9]_P = (\bar{p}_2-\bar{p}_1)\bar{p}_8,\\
&[\bar{p}_3,\bar{p}_{10}]_P = \bar{p}_6\bar{p}_7 - \bar{p}_4\bar{p}_8, [\bar{p}_4,\bar{p}_5]_P = -4\bar{p}_0\bar{p}_6,\ [\bar{p}_4,\bar{p}_6]_P = -2\bar{p}_0\bar{p}_4,\\
&[\bar{p}_4,\bar{p}_7]_P = (\bar{p}_1+\bar{p}_2)\bar{p}_4,\ [\bar{p}_4,\bar{p}_8]_P = (\bar{p}_1+\bar{p}_2)\bar{p}_6 - 2\bar{p}_0\bar{p}_7,\\
&[\bar{p}_4,\bar{p}_9]_P = 2(\bar{p}_1+\bar{p}_2)\bar{p}_7,\ [\bar{p}_4,\bar{p}_{10}]_P = 0,\ [\bar{p}_5,\bar{p}_6]_P = 2\bar{p}_0\bar{p}_5,\\
&[\bar{p}_5,\bar{p}_7]_P = 2\bar{p}_3\bar{p}_6 + 2\bar{p}_0\bar{p}_8,\ [\bar{p}_5,\bar{p}_8]_P = 2\bar{p}_3\bar{p}_5,\ [\bar{p}_5,\bar{p}_9]_P = 4\bar{p}_3\bar{p}_8,\ [\bar{p}_5,\bar{p}_{10}]_P = 0,\\
&[\bar{p}_6,\bar{p}_7]_P = \frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_6 + \bar{p}_3\bar{p}_4 + \bar{p}_0\bar{p}_7,\ [\bar{p}_6,\bar{p}_8]_P = \frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_5 + \bar{p}_3\bar{p}_6\\
&\qquad - \bar{p}_0\bar{p}_8,\ [\bar{p}_6,\bar{p}_9]_P = (\bar{p}_1+\bar{p}_2)\bar{p}_8 + 2\bar{p}_3\bar{p}_7,\ [\bar{p}_6,\bar{p}_{10}]_P = 0,\\
&[\bar{p}_7,\bar{p}_8]_P = \frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_8 - \bar{p}_3\bar{p}_7 - \bar{p}_0(\bar{p}_9+\bar{p}_{10}),\ [\bar{p}_7,\bar{p}_9]_P = (\bar{p}_1+\bar{p}_2)(\bar{p}_9+\bar{p}_{10}),\\
&[\bar{p}_7,\bar{p}_{10}]_P = -\frac{1}{2}(\bar{p}_2+\bar{p}_1)\bar{p}_6^2 - \bar{p}_0\bar{p}_6\bar{p}_7 + \bar{p}_0\bar{p}_4\bar{p}_8 + \frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_4\bar{p}_5,\\
&[\bar{p}_8,\bar{p}_9]_P = 2\bar{p}_3(\bar{p}_9+\bar{p}_{10}), [\bar{p}_8,\bar{p}_{10}]_P = \bar{p}_0\bar{p}_6\bar{p}_8 - \bar{p}_3\bar{p}_6^2 - \bar{p}_0\bar{p}_5\bar{p}_7 + \bar{p}_3\bar{p}_4\bar{p}_5,\\
&[\bar{p}_9,\bar{p}_{10}]_P = (\bar{p}_5\bar{p}_7 - \bar{p}_6\bar{p}_8)(\bar{p}_1+\bar{p}_2) + 2\bar{p}_3\bar{p}_4\bar{p}_8 - 2\bar{p}_3\bar{p}_6\bar{p}_7\ .
\end{aligned}$$

Relation (3.18) is transformed into

$$\bar{p}_{10}^2 - \bar{p}_4\bar{p}_5\bar{p}_9 - 2\bar{p}_6\bar{p}_7\bar{p}_8 + \bar{p}_9\bar{p}_6^2 + \bar{p}_4\bar{p}_8^2 + \bar{p}_5\bar{p}_7^2 = 0 \,.$$

Additional relation (3.19) in the case $n = 2$ is again as for $Q = \mathbf{P}^n(\mathbb{H})$

$$\bar{p}_1\bar{p}_2 - \bar{p}_3^2 - \bar{p}_9 = 0 \,, \tag{7.17}$$

and can be used in this case for excluding $\bar{p}_9$ from the list of generators.

All Poisson algebras $\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{H})_\mathbf{S}\right)$ for $n \geqslant 3$ are isomorphic to each other. The same is also valid for the Hamiltonian function.

Elements $\overline{C}_i^{\mathrm{gr}}$ from $\mathrm{Z}\,\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{H})_\mathbf{S}\right)$, $i = 1,2,3$, corresponding to elements $\overline{C}_i$, $i = 1,2,3$ from Sect. 3.2.2, are:

$$\begin{aligned}
\overline{C}_1^{\mathrm{gr}} &= \bar{p}_0^2 + \bar{p}_1 - \bar{p}_2 + \bar{p}_4 - \bar{p}_5,\ \overline{C}_2^{\mathrm{gr}} = \bar{p}_1\bar{p}_2 - \bar{p}_3^2 - \bar{p}_9,\ \overline{C}_3^{\mathrm{gr}} = \frac{1}{2}(\bar{p}_1 - \bar{p}_2)(\bar{p}_4 - \bar{p}_5) \\
&+ \frac{1}{4}(\bar{p}_1 + \bar{p}_2)^2 - \bar{p}_3^2 + \frac{1}{4}(\bar{p}_4 + \bar{p}_5)^2 - \bar{p}_6^2 - \bar{p}_9 + 2\bar{p}_{10} \\
&+ \frac{1}{2}\bar{p}_0^2(\bar{p}_1 - \bar{p}_2 + \bar{p}_4 - \bar{p}_5) + \frac{1}{4}\bar{p}_0^4 \,.
\end{aligned}$$

Note that due to (7.17) it holds $\overline{C}_2^{\mathrm{gr}} = 0$ for $n = 2$.

7.2.2 Octonionic Case

The expression for the two-body Hamiltonian function for $Q = \mathbf{H}^2(\mathbb{C}a)_\mathbf{S}$ coincides with (7.15), where $\bar{p}_i$, $i = 0,\dots,9$ are now generators of the Poisson algebra

$$\mathcal{P}_I\left(\mathbf{H}^2(\mathbb{C}a)_\mathbf{S}\right) \cong \mathrm{gr}\,\mathrm{Diff}_I\left(\mathbf{H}^2(\mathbb{C}a)_\mathbf{S}\right),$$

corresponding to generators $\bar{D}_i$, $i = 0,\dots,9$ of the algebra $\mathrm{Diff}_I\left(\mathbf{H}^2(\mathbb{C}a)_\mathbf{S}\right)$.

Due to (3.45) commutative relations for them are

$$\begin{aligned}
&[\bar{p}_0,\bar{p}_1]_P = \bar{p}_3,\ [\bar{p}_0,\bar{p}_2]_P = \bar{p}_3,\ [\bar{p}_0,\bar{p}_3]_P = \frac{1}{2}(\bar{p}_1 + \bar{p}_2),\ [\bar{p}_0,\bar{p}_4]_P = 2\bar{p}_6, \\
&[\bar{p}_0,\bar{p}_5]_P = 2\bar{p}_6,\ [\bar{p}_0,\bar{p}_6]_P = \bar{p}_4 + \bar{p}_5,\ [\bar{p}_0,\bar{p}_7]_P = \bar{p}_8,\ [\bar{p}_0,\bar{p}_8]_P = \bar{p}_7,\ [\bar{p}_0,\bar{p}_9]_P = 0, \\
&[\bar{p}_1,\bar{p}_2]_P = -2\bar{p}_0\bar{p}_3 - 2\bar{p}_7,\ [\bar{p}_1,\bar{p}_3]_P = -\bar{p}_0\bar{p}_1 - \bar{p}_8,\ [\bar{p}_1,\bar{p}_4]_P = -2\bar{p}_7, \\
&[\bar{p}_1,\bar{p}_5]_P = 0,\ [\bar{p}_1,\bar{p}_6]_P = -\bar{p}_8,\ [\bar{p}_1,\bar{p}_7]_P = -\bar{p}_1(\bar{p}_2 + \bar{p}_4) + \bar{p}_9 + \bar{p}_3\bar{p}_6 + \bar{p}_3^2, \\
&[\bar{p}_1,\bar{p}_8]_P = \bar{p}_3\bar{p}_5 - \bar{p}_1\bar{p}_6,\ [\bar{p}_1,\bar{p}_9]_P = \bar{p}_6\bar{p}_8 - \bar{p}_5\bar{p}_7, \\
&[\bar{p}_2,\bar{p}_3]_P = \bar{p}_0\bar{p}_2 + \bar{p}_8,\ [\bar{p}_2,\bar{p}_4]_P = -2\bar{p}_7,\ [\bar{p}_2,\bar{p}_5]_P = 0,\ [\bar{p}_2,\bar{p}_6]_P = -\bar{p}_8, \\
&[\bar{p}_2,\bar{p}_7]_P = (\bar{p}_4 - \bar{p}_1)\bar{p}_2 + \bar{p}_9 - \bar{p}_3\bar{p}_6 + \bar{p}_3^2,\ [\bar{p}_2,\bar{p}_8]_P = \bar{p}_2\bar{p}_6 - \bar{p}_3\bar{p}_5, \\
&[\bar{p}_2,\bar{p}_9]_P = \bar{p}_6\bar{p}_8 - \bar{p}_5\bar{p}_7,\ [\bar{p}_3,\bar{p}_4]_P = 0,\ [\bar{p}_3,\bar{p}_5]_P = 2\bar{p}_8,\ [\bar{p}_3,\bar{p}_6]_P = \bar{p}_7, \\
&[\bar{p}_3,\bar{p}_7]_P = \frac{1}{2}(\bar{p}_2 - \bar{p}_1)\bar{p}_6, \\
&[\bar{p}_3,\bar{p}_8]_P = \bar{p}_1\bar{p}_2 + \frac{1}{2}(\bar{p}_2 - \bar{p}_1)\bar{p}_5 - \bar{p}_9 - \bar{p}_3^2,\ [\bar{p}_3,\bar{p}_9]_P = \bar{p}_4\bar{p}_8 - \bar{p}_6\bar{p}_7, \qquad (7.18) \\
&[\bar{p}_4,\bar{p}_5]_P = -4\bar{p}_0\bar{p}_6,\ [\bar{p}_4,\bar{p}_6]_P = -2\bar{p}_0\bar{p}_4,\ [\bar{p}_4,\bar{p}_7]_P = (\bar{p}_1 + \bar{p}_2)\bar{p}_4,
\end{aligned}$$

$$[\bar{p}_4,\bar{p}_8]_P=(\bar{p}_1+\bar{p}_2)\bar{p}_6-2\bar{p}_0\bar{p}_7,\ [\bar{p}_4,\bar{p}_9]_P=0,\ [\bar{p}_5,\bar{p}_6]_P=2\bar{p}_0\bar{p}_5,$$
$$[\bar{p}_5,\bar{p}_7]_P=2\bar{p}_3\bar{p}_6+2\bar{p}_0\bar{p}_8, [\bar{p}_5,\bar{p}_8]_P=2\bar{p}_3\bar{p}_5,\ [\bar{p}_5,\bar{p}_9]_P=0,$$
$$[\bar{p}_6,\bar{p}_7]_P=\frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_6+\bar{p}_3\bar{p}_4+\bar{p}_0\bar{p}_7,$$
$$[\bar{p}_6,\bar{p}_8]_P=\frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_5+\bar{p}_3\bar{p}_6-\bar{p}_0\bar{p}_8,\ [\bar{p}_6,\bar{p}_9]_P=0,$$
$$[\bar{p}_7,\bar{p}_8]_P=\frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_8-\bar{p}_3\bar{p}_7+\bar{p}_0\bar{p}_9+\bar{p}_0\bar{p}_3^2-\bar{p}_0\bar{p}_1\bar{p}_2,$$
$$[\bar{p}_7,\bar{p}_9]_P=-\frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_4\bar{p}_5+\bar{p}_0\bar{p}_6\bar{p}_7-\bar{p}_0\bar{p}_4\bar{p}_8+\frac{1}{2}(\bar{p}_1+\bar{p}_2)\bar{p}_6^2,$$
$$[\bar{p}_8,\bar{p}_9]_P=\bar{p}_3\bar{p}_6^2-\bar{p}_0\bar{p}_6\bar{p}_8-\bar{p}_3\bar{p}_4\bar{p}_5+\bar{p}_0\bar{p}_5\bar{p}_7\ .$$

Elements $\overline{C}_i^{\mathrm{gr}}$, $i=1,2$ from $\mathrm{Z}\,\mathcal{P}_I\left(\mathbf{H}^2(\mathbb{C}a)_{\mathbf{S}}\right)$, corresponding to elements $\overline{C}_1,\overline{C}_2$ from Sect. 3.5.2, have the form

$$\overline{C}_1^{\mathrm{gr}}=\bar{p}_0^2+\bar{p}_1-\bar{p}_2+\bar{p}_4-\bar{p}_5,\ \overline{C}_2^{\mathrm{gr}}=\bar{p}_4\bar{p}_5-\bar{p}_6^2-2\bar{p}_9\ .$$

7.2.3 Complex Case

From expression (5.26) one gets for $Q=\mathbf{H}^n(\mathbb{C})$ the two-body Hamiltonian function

$$\begin{aligned}h=&\frac{(1-r^2)^2}{8mR^2}p_r^2+\frac{(m_1\alpha-m_2\beta)(1-r^2)}{2m_1m_2R^2}p_r\bar{p}_0+\frac{m_1\alpha^2+m_2\beta^2}{2m_1m_2R^2}\bar{p}_0^2\\&+\frac{1}{2}\left(D_h\bar{p}_1+F_h\bar{p}_2+2E_h\bar{p}_3+C_h\bar{p}_4^2+A_h\bar{p}_5^2+2B_h\bar{p}_4\bar{p}_5\right)+V(r),\end{aligned}\tag{7.19}$$

where $\bar{p}_i$, $i=0,\dots,5$ are generators of the Poisson algebra $\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}}\right)\cong\mathrm{gr}\,\mathrm{Diff}_I\left(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}}\right)$, corresponding to generators $\bar{D}_i$, $i=0,\dots,5$ of the algebra $\mathrm{Diff}_I\left(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}}\right)$. Denote the last generators of $\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}}\right)$, corresponding to the generator $\bar{\Box}$ of $\mathrm{Diff}_I\left(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}}\right)$, by $\bar{p}_\Box$.

The commutative relations for these generators of $\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}}\right)$ are as follows

$$\begin{aligned}&[\bar{p}_0,\bar{p}_1]_P=\bar{p}_3,\ [\bar{p}_0,\bar{p}_2]_P=\bar{p}_3,\ [\bar{p}_0,\bar{p}_3]_P=\frac{1}{2}(\bar{p}_1+\bar{p}_2),\ [\bar{p}_0,\bar{p}_4]_P=\bar{p}_5,\\&[\bar{p}_0,\bar{p}_5]_P=\bar{p}_4, [\bar{p}_0,\bar{p}_\Box]_P=0,\ [\bar{p}_1,\bar{p}_2]_P=-2\bar{p}_0\bar{p}_3-2\bar{p}_\Box\bar{p}_4,\\&[\bar{p}_1,\bar{p}_3]_P=-\bar{p}_0\bar{p}_1-\bar{p}_\Box\bar{p}_5,\ [\bar{p}_1,\bar{p}_4]_P=-\bar{p}_\Box,\\&[\bar{p}_1,\bar{p}_5]_P=0,\ [\bar{p}_1,\bar{p}_\Box]_P=\bar{p}_3\bar{p}_5-\bar{p}_1\bar{p}_4,\ [\bar{p}_2,\bar{p}_3]_P=\bar{p}_0\bar{p}_2+\bar{p}_\Box\bar{p}_5,\\&[\bar{p}_2,\bar{p}_4]_P=-\bar{p}_\Box, [\bar{p}_2,\bar{p}_5]_P=0,\ [\bar{p}_2,\bar{p}_\Box]_P=\bar{p}_2\bar{p}_4-\bar{p}_3\bar{p}_5,\\&[\bar{p}_3,\bar{p}_4]_P=0,\ [\bar{p}_3,\bar{p}_5]_P=\bar{p}_\Box,\ [\bar{p}_3,\bar{p}_\Box]_P=\frac{1}{2}(\bar{p}_2-\bar{p}_1)\bar{p}_5,\\&[\bar{p}_4,\bar{p}_5]_P=-\bar{p}_0,\ [\bar{p}_4,\bar{p}_\Box]_P=\frac{1}{2}(\bar{p}_1+\bar{p}_2),\ [\bar{p}_5,\bar{p}_\Box]_P=\bar{p}_3\ .\end{aligned}\tag{7.20}$$

The additional relation (3.29) in the case $n = 2$ now becomes

$$\bar{p}_1\bar{p}_2 - \bar{p}_3^2 - \bar{p}_\square^2 = 0\ . \tag{7.21}$$

Elements $\overline{C}_i^{\mathrm{gr}} \in \mathrm{Z}\,\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{C})_{\mathbf{S}}\right),\ i = 1,2,3$, corresponding to elements $\overline{C}_i$, $i = 1,2,3$ from Sect. 3.3.2, are

$$\overline{C}_1^{\mathrm{gr}} = \bar{p}_0^2 + \bar{p}_1 - \bar{p}_2 + \bar{p}_4^2 - \bar{p}_5^2,\ \overline{C}_2^{\mathrm{gr}} = (\bar{p}_1 + \bar{p}_2)\bar{p}_5 - 2\bar{p}_3\bar{p}_4 + 2\bar{p}_0\bar{p}_\square,$$
$$\overline{C}_3^{\mathrm{gr}} = \bar{p}_1\bar{p}_2 - \bar{p}_3^2 - \bar{p}_\square^2\ .$$

Note that due to (7.21) it holds $\overline{C}_3^{\mathrm{gr}} = 0$ for $n = 2$.

7.2.4 Real Case

From expression (5.28) one gets for $Q = \mathbf{H}^n(\mathbb{R})$, $n \geqslant 3$ the two-body Hamiltonian function in the form

$$\begin{aligned} h = {} & \frac{(1-r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1-r^2)}{2m_1m_2R^2}p_r\bar{p}_0 + \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}\bar{p}_0^2 \\ & + \frac{1}{2}\left(C_h\bar{p}_1 + A_h\bar{p}_2 + 2B_h\bar{p}_3\right) + V(r)\ . \end{aligned} \tag{7.22}$$

The Poisson algebra $\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{R})_{\mathbf{S}}\right)$ is generated by elements $\bar{p}_i$, $i = 0,\ldots,3$ for $n \geqslant 4$ and by elements $\bar{p}_\square, \bar{p}_i$, $i = 0,\ldots,3$ for $n = 3$. The commutative relations for these generators are

$$\begin{aligned} & [\bar{p}_0,\bar{p}_1]_P = 2\bar{p}_3,\ [\bar{p}_0,\bar{p}_2]_P = 2\bar{p}_3,\ [\bar{p}_0,\bar{p}_3]_P = \bar{p}_1 + \bar{p}_2, \\ & [\bar{p}_1,\bar{p}_2]_P = -4\bar{p}_0\bar{p}_3,\ [\bar{p}_1,\bar{p}_3]_P = -2\bar{p}_0\bar{p}_1,\ [\bar{p}_2,\bar{p}_3]_P = 2\bar{p}_0\bar{p}_2\ . \end{aligned} \tag{7.23}$$

The generator $\bar{p}_\square$, if exists, commutes with all other generators.

The additional relation (3.37) in the case $n = 3$ now becomes

$$\bar{p}_1\bar{p}_2 - \bar{p}_3^2 - \bar{p}_\square^2 = 0\ . \tag{7.24}$$

Elements $\overline{C}_i^{\mathrm{gr}} \in \mathrm{Z}\,\mathcal{P}_I\left(\mathbf{H}^n(\mathbb{R})_{\mathbf{S}}\right),\ i = 1,2$, corresponding to elements $\overline{C}_i$, $i = 1,2$ from Sect. 3.4.2, are

$$\overline{C}_1^{\mathrm{gr}} = \bar{p}_0^2 + \bar{p}_1 - \bar{p}_2,\ \overline{C}_2^{\mathrm{gr}} = \bar{p}_1\bar{p}_2 - \bar{p}_3^2\ .$$

Note that due to (7.12) it holds $\overline{C}_2^{\mathrm{gr}} = \bar{p}_\square^2$ for $n = 3$.

The expression (5.29) implies for $Q = \mathbf{H}^2(\mathbb{R})$:

$$\begin{aligned} h = {} & \frac{(1-r^2)^2}{8mR^2}p_r^2 + \frac{(m_1\alpha - m_2\beta)(1-r^2)}{2m_1m_2R^2}p_r\bar{p}_0 + \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}\bar{p}_0^2 \\ & + \frac{1}{2}\left(C_h\bar{p}_1^2 + A_h\bar{p}_2^2 + 2B_h\bar{p}_1\bar{p}_2\right) + V(r), \end{aligned}$$

where

$$[\bar{p}_0,\bar{p}_1]_P = \bar{p}_2,\ [\bar{p}_0,\bar{p}_2]_P = \bar{p}_1,\ [\bar{p}_1,\bar{p}_2]_P = -\bar{p}_0\ . \tag{7.25}$$

The Poisson algebra $\mathcal{P}_I\left(\mathbf{H}^2(\mathbb{R})_{\mathbf{S}}\right)$ is isomorphic to the graded algebra $\mathrm{gr}\,U(\mathfrak{so}\ (1,2))$. There is only one central independent element $\overline{C}_1^{\mathrm{gr}} = \bar{p}_0^2 + \bar{p}_1^2 - \bar{p}_2^2$ in this algebra and this is the simplest noncompact case.

7.3 Dynamics of the Two-Body System and the Problem of Particles' Collision

Note first of all that the classical two-body problem on spaces $\mathbf{P}^n(\mathbb{H}), \mathbf{P}^n(\mathbb{C})$, $\mathbf{P}^n(\mathbb{R}), \mathbf{S}^n$ and their hyperbolic analogs $\mathbf{H}^n(\mathbb{H})$, $\mathbf{H}^n(\mathbb{C})$, $\mathbf{H}^n(\mathbb{R})$ reaches its full generality at $n = 3$, since for $n > 3$ there is some totally geodesic subspace $\widetilde{Q}$, isometric respectively to $\mathbf{P}^3(\mathbb{H})$, $\mathbf{P}^3(\mathbb{C})$, $\mathbf{P}^3(\mathbb{R})$, $\mathbf{S}^3$, $\mathbf{H}^3(\mathbb{H})$, $\mathbf{H}^3(\mathbb{C})$, $\mathbf{H}^3(\mathbb{R})$, such that it contains initial positions of particles and their initial velocities are tangent to $\widetilde{Q}$. This occurs in accordance with the stabilization of Hamiltonian functions (7.3), (7.7), (7.10), (7.15), (7.19), (7.22) and corresponding Poisson algebras for $n \geqslant 3$.

It is known that Hamiltonian mechanics admits the following approach, which is in close connection with quantum mechanics. Under this approach smooth functions on a symplectic phase space M are called *observables*. One is interested in their evolution w.r.t. a phase flow φ_t^h, corresponding to a Hamiltonian function h:

$$f_t(x) := f(\varphi_t^h(x)),\ \frac{d\varphi_t^h}{dt} = X_h,\ x \in M,\ f \in C^\infty(M)\ .$$

Due to (4.4) the evolution of an observable f_t is described by the equation

$$\frac{df_t}{dt} = [f_t, h]_P\ . \tag{7.26}$$

An integral of the flow φ_t^h is a constant observable.

Let now a Lie group G acts in a Poisson way on the space M. One can consider only G-invariant observables and particularly only G-invariant Hamiltonian functions. In fact, this program for the two-body problem on two-point homogeneous space was realized in Sects. 7.1 and 7.2. Indeed, there were found the full system of independent G-invariant observables $r, p_r, p_0, p_1, \ldots$ on the corresponding cotangent bundle, the expression of the two-body Hamiltonian function through this system and commutator relations (7.4), (7.6), (7.8), (7.11), (7.14), (7.16), (7.18), (7.20), (7.23), (7.25). Together with obvious commutator relations

$$[r, p_r]_P = 1,\ [r, p_i]_P = 0,\ [p_r, p_i]_P = 0,\ i = 0, 1, 2, \ldots$$

this allows one to write (7.26) in an explicit form for the base of G-invariant observables.

Evidently, the functions C_i^{gr} and $\overline{C}_i^{\mathrm{gr}}$ from above are integrals of the two-body motion for any central potential $V(r)$.

Remark 7.1. *The construction of* Ad_{K_0}*-invariant elements in* $S(\widetilde{\mathfrak{p}})$ *(Chap. 3), Remark* 4.3 *and the Cauchy inequality in real, complex or quaternion vector spaces imply the following inequalities for functions* p_i *and* $\bar{p}_i$ *on the phase space.*

1. *For spaces* $\mathbf{P}^n(\mathbb{H}), \mathbf{H}^n(\mathbb{H})$*:*

$$p_1, p_2, p_4, p_5, p_9, \bar{p}_1, \bar{p}_2, \bar{p}_4, \bar{p}_5, \bar{p}_9 \geqslant 0,$$
$$p_1p_2 - p_3^2 - p_9 \geqslant 0,\ p_4p_5 - p_6^2 \geqslant 0,\ \bar{p}_1\bar{p}_2 - \bar{p}_3^2 - \bar{p}_9 \geqslant 0,\ \bar{p}_4\bar{p}_5 - \bar{p}_6^2 \geqslant 0,$$
$$p_7^2 \leqslant p_4p_9,\ p_8^2 \leqslant p_5p_9,\ \bar{p}_7^2 \leqslant \bar{p}_4\bar{p}_9,\ \bar{p}_8^2 \leqslant \bar{p}_5\bar{p}_9 .$$

2. For spaces $\mathbf{P}^n(\mathbb{C}), \mathbf{H}^n(\mathbb{C})$*:*

$$p_1, p_2, \bar{p}_1, \bar{p}_2 \geqslant 0, p_1p_2 - p_3^2 - p_\square^2 \geqslant 0,\ \bar{p}_1\bar{p}_2 - \bar{p}_3^2 - \bar{p}_\square^2 \geqslant 0 .$$

3. For spaces $\mathbf{P}^n(\mathbb{R}), \mathbf{S}^n, \mathbf{H}^n(\mathbb{R})$*:*

$$p_1, p_2, \bar{p}_1, \bar{p}_2 \geqslant 0, p_1p_2 - p_3^2 \geqslant 0,\ \bar{p}_1\bar{p}_2 - \bar{p}_3^2 \geqslant 0 .$$

4. For spaces $\mathbf{P}^2(\mathbb{C}a), \mathbf{H}^2(\mathbb{C}a)$*:*

$$p_1, p_2, p_4, p_5, \bar{p}_1, \bar{p}_2, \bar{p}_4, \bar{p}_5 \geqslant 0, p_1p_2 - p_3^2 \geqslant 0,\ \bar{p}_1\bar{p}_2 - \bar{p}_3^2 \geqslant 0,\ p_4p_5 - p_6^2 \geqslant 0,$$
$$\bar{p}_4\bar{p}_5 - \bar{p}_6^2 \geqslant 0,\ |p_7| \leqslant 14\sqrt{p_1p_2p_4},\ |\bar{p}_7| \leqslant 14\sqrt{\bar{p}_1\bar{p}_2\bar{p}_4},\ |p_8| \leqslant 14\sqrt{p_1p_2p_5},$$
$$|\bar{p}_8| \leqslant 14\sqrt{\bar{p}_1\bar{p}_2\bar{p}_5},\ |p_9| \leqslant 16 \cdot 49\sqrt{p_1p_2p_4p_5},\ |\bar{p}_9| \leqslant 16 \cdot 49\sqrt{\bar{p}_1\bar{p}_2\bar{p}_4\bar{p}_5} .$$

7.3.1 The Problem of Particles' Collision

The system of ordinary differential equations for variables $r, p_r, p_0, p_1, \ldots$ determines, in particular, the evolution of the distance between particles. If the potential V is a smooth function on the space $(Q \times Q) \setminus \mathrm{diag}$, bounded when the distance between particles tends to infinity (for a noncompact space Q), then the only obstruction to the existence of the global solution for this dynamical system is the possible collision of particles. For a repulsive potential $V(r)$ such that $V(r) \to +\infty$ as $r \to 0$ a collision can not occur due to the energy conservation. Consider an attractive potential $V(r)$.

There are two possible scenarios of a particles collision at a moment t_0. The first one, which is quite natural, corresponds to $\lim_{t\to t_0-0} r(t) = 0$. The second scenario is more pathological: $\liminf_{t\to t_0-0} r(t) = 0$, but $\lim_{t\to t_0-0} r(t)$ does not exist.

Suppose that for any $\varepsilon > 0$ it holds

$$V(r) \geqslant C_1(\varepsilon) = \mathrm{const},\ \|\operatorname{grad} V(r)\| \geqslant C_2(\varepsilon) = \mathrm{const},\ \forall r \geqslant \varepsilon . \tag{7.27}$$

Using arguments from [140] (see also [151]) one can prove that the second scenario is impossible for such potentials.[2] Indeed, let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0 \tag{7.28}$$

be a system of ordinary differential equations, corresponding to the two-body problem on a two-point homogeneous space Q w.r.t. local coordinates. The standard existence theorem ([39], Chap. 2) guarantees the existence of the

[2] The consideration of Painlevé concerns the n-body problem ($n \geqslant 3$) with Newtonian interactions in Euclidean space. Denote in this case $r(t) := \min_{i\neq j} r_{ij}(t)$, where r_{ij} is the distance between ith and jth particles. Painlevé proved that if it holds $\liminf_{t\to t_0-0} r(t) = 0$, then also $\lim_{t\to t_0-0} r(t) = 0$.

solution of (7.28) in a time interval of length determined by an upper bound on $\|\mathbf{f}(\mathbf{x})\|$ in some domain, containing $\mathbf{x}(0)$. The energy conservation law and inequalities (7.27) leads to the boundedness of particles' velocities and thus to the boundedness of $\|\mathbf{f}(\mathbf{x})\|$ for $r \geqslant \varepsilon > 0$. Therefore, the condition $r(t') \geqslant \varepsilon > 0$ implies the existence of a solution for the two-body problem on Q (and thus the inequality $r(t) > 0$) for $t' < t < t + T_\varepsilon$, where $T_\varepsilon > 0$ depends only upon ε. Now the assumptions $\liminf_{t \to t_0 - 0} r(t) = 0$ and $r(t_k) > \varepsilon > 0$ for a infinite sequence $t_k \to t_0 - 0$ leads to a contradiction for $t_0 - t_k < T_\varepsilon$. □

In Euclidean case the critical asymptotic as $r \to 0$ is the centrifugal potential

$$V \sim \frac{c}{r^2},\ c < 0.$$

Thus, for potentials $V = o(r^{-2})$ there are no collisions for general initial conditions, which do not correspond to the frontal motion of particles. For curved two-point homogeneous spaces the situation is more subtle.

One can separate initial conditions for the two-body system by values of the corresponding integrals of motion C_i^{gr} and $\overline{C}_i^{\mathrm{gr}}$. Some of these values may correspond to the frontal motion of particles along a common geodesic that leads to their collision. On the other hand, the same values may correspond to a more complicated motion. In the second case, due to the possible nonintegrability of the two-body problem, it is difficult to guarantee the absence of collisions.

Other values of integrals cannot correspond to the frontal motion of particles along a common geodesic. For these values the theorem below guarantees the absence of particles collision. It generalizes the result, obtained for spaces of a constant curvature in [160], for all two-point homogeneous spaces.

Choose for the simplicity $\alpha = m_2/(m_1 + m_2), \beta = m_1/(m_1 + m_2)$. In this case the functions $A_{s,h}, B_{s,h}, C_{s,h}, D_{s,h}, E_{s,h}, F_{s,h}$ have following asymptotics as $r \to 0$:

$$A_{s,h}(r) = \frac{1}{4mR^2r^2} + O(1),\ B_{s,h}(r) = O(r),\ C_{s,h}(r) = \frac{1}{(m_1 + m_2)R^2} + O(r^2),$$
$$D_{s,h}(r) = \frac{1}{(m_1 + m_2)R^2} + O(r^2),\ E_{s,h}(r) = O(r),\ F_{s,h}(r) = \frac{1}{mR^2r^2} + O(1)\,. \tag{7.29}$$

Theorem 7.1. *Suppose that the potential $V(r)$ is smooth at $r > 0$, for any $\varepsilon > 0$, inequalities (7.27) are valid for some real constants $C_1(\varepsilon), C_2(\varepsilon)$ and $V = o(r^{-2})$ as $r \to 0$. Then the collision of particles will not happen in the following cases (everywhere $c_i = \mathrm{const}$):*

1. $Q = \mathbf{P}^n(\mathbb{H}),\ n \geqslant 3,\ C_2^{\mathrm{gr}} = c_2 > 0\,;$
2. $Q = \mathbf{P}^n(\mathbb{C}),\ n \geqslant 3,\ C_3^{\mathrm{gr}} = c_3 > 0\,;$
3. $Q = \mathbf{P}^n(\mathbb{R}), \mathbf{S}^n,\ n \geqslant 3,\ C_2^{\mathrm{gr}} = c_2 > 0\,;$
4. $Q = \mathbf{H}^n(\mathbb{H})$:
   a) $n \geqslant 3,\ \overline{C}_2^{\mathrm{gr}} = c_2 > 0\,;$
   b) $n \geqslant 2,\ \overline{C}_1^{\mathrm{gr}} = c_1 < 0\,;$
   c) $n \geqslant 2,\ \overline{C}_1^{\mathrm{gr}} = 0$ *and* $\left(\bar{p}_0^2 + \bar{p}_1 + \bar{p}_2 + \bar{p}_4 + \bar{p}_5\right)\big|_{t=0} > 0\,;$

5. $Q = \mathbf{H}^n(\mathbb{C})$:
   a) $n \geqslant 3$, $\overline{C}_3^{\rm gr} = c_3 > 0$;
   b) $n \geqslant 2$, $\overline{C}_1^{\rm gr} = c_1 < 0$;
   c) $n \geqslant 2$, $\overline{C}_1^{\rm gr} = 0$ *and* $\left(\bar{p}_0^2 + \bar{p}_1 + \bar{p}_2 + \bar{p}_4 + \bar{p}_5\right)\big|_{t=0} > 0$;
6. $Q = \mathbf{H}^n(\mathbb{R})$:
   a) $n \geqslant 3$, $\overline{C}_2^{\rm gr} = c_2 > 0$;
   b) $n \geqslant 2$, $\overline{C}_1^{\rm gr} = c_1 < 0$;
   c) $n \geqslant 2$, $\overline{C}_1^{\rm gr} = 0$ *and* $\left(\bar{p}_0^2 + \bar{p}_1 + \bar{p}_2\right)\big|_{t=0} > 0$;
7. $Q = \mathbf{H}^2(\mathbb{C}a)$:
   a) $\overline{C}_1^{\rm gr} = c_1 < 0$;
   b) $\overline{C}_1^{\rm gr} = 0$ *and* $\left(\bar{p}_0^2 + \bar{p}_1 + \bar{p}_2 + \bar{p}_2 + \bar{p}_4 + \bar{p}_5\right)\big|_{t=0} > 0$.

Proof. We shall carry this proof by *reductio ad absurdum*. Due to the discussion above one can suppose that some trajectory of the two-body system, started at $t = 0$, is regular as $0 < t < t_0$ and $\lim_{t\to t_0-0} r(t) \to 0$ for some $t_0 > 0$. Note first of all that inequalities from Remark 7.1, asymptotics (7.29) and the energy conservation law imply $p_2(t), p_5(t) \to 0$ or $\bar{p}_2(t), \bar{p}_5(t) \to 0$ as $t \to t_0 - 0$. Therefore, the equalities $C_1^{\rm gr} = \text{const}$, $\overline{C}_1^{\rm gr} = \text{const}$ and inequalities from Remark 7.1 again imply that all functions $p_i, \bar{p}_i$ are bounded as $0 \leqslant t < t_0$ on this trajectory.

Consider the case 4(a). Here, $\bar{p}_1\bar{p}_2 = c_2 + \bar{p}_3^2 + \bar{p}_9 \geqslant c_2$ and therefore $\bar{p}_1 \geqslant c_2/\bar{p}_2 \to +\infty$, $t \to t_0 - 0$ that contradicts to $\bar{p}_1 \leqslant c_1 + \bar{p}_2 + \bar{p}_5 \to c_1$, $t \to t_0 - 0$. Proofs in cases 1, 2, 3, 5(a) and 6(a) are completely similar.

In the case 4(b) one has $\bar{p}_0^2 + \bar{p}_1 + \bar{p}_4 = c_1 + \bar{p}_2 + \bar{p}_5 \to c_1 < 0$, $t \to t_0 - 0$ that contradicts to inequalities from Remark 7.1. Cases 5(b), 6(b) and 7(a) are completely similar.

In the case 7(b) one has

$$\bar{p}_0^2 = \bar{p}_2 + \bar{p}_5 - \bar{p}_1 - \bar{p}_4 \leqslant \bar{p}_2 + \bar{p}_5 \,. \tag{7.30}$$

Together with inequalities from Remark 7.1 this means that conditions $\bar{p}_2 = \bar{p}_5 = 0$ imply $\bar{p}_i = 0$, $i = 0, \ldots, 9$. Thus, it holds $(\bar{p}_2 + \bar{p}_5)|_{t=0} > 0$. Therefore, without loss of generality, one can additionally suppose that t_0 is the minimal positive value of t such that

$$\lim_{t\to t_0-0} (\bar{p}_2 + \bar{p}_5) = 0 \,.$$

From (7.15), (7.18), (7.26) one gets:

$$\frac{d}{dt}\,(\bar{p}_2 + \bar{p}_5) = -\frac{\bar{p}_0(\bar{p}_3 + 2\bar{p}_6)}{(m_1+m_2)R^2} + D_h(\bar{p}_0\bar{p}_3 + \bar{p}_7) + E_h(\bar{p}_0\bar{p}_2 - \bar{p}_8) + C_h(-\bar{p}_7 + 2\bar{p}_0\bar{p}_6) + B_h(-\bar{p}_8 + 2\bar{p}_0\bar{p}_5) \,.$$

Therefore, equality (7.30) and Remark 7.1 imply the following estimate

$$\left|\frac{d}{dt}\,(\bar{p}_2 + \bar{p}_5)\right| \leqslant C\,(\bar{p}_2 + \bar{p}_5)\,,$$

in an ε-neighborhood of the value t_0 for some $\varepsilon > 0$ and $C = \text{const} > 0$. In view of the Gronwall inequality this estimate means that

$$|\bar{p}_2(t) + \bar{p}_5(t) - \bar{p}_2(t_0 - 0) - \bar{p}_5(t_0 - 0)| \leqslant (\bar{p}_2(t_0 - 0) + \bar{p}_5(t_0 - 0)) \exp(C(t_0 - t)) = 0,$$

for $t_0 - \varepsilon < t < t_0$. This implies the equality $\bar{p}_2(t) + \bar{p}_5(t) = 0$ for some positive $t < t_0$ that contradicts to the minimality of t_0. Proofs in cases 4(c), 5(c) and 6(c) are similar. □

Remark 7.2. *Informally, conditions of Theorem 7.1 mean that at a small distance the rotation motion of two particles dominates over their translational motion. In all cases of this theorem except 4(c), 5(c), 6(c) and 7(b) it is not difficult to find for a concrete potential V, obeying the condition $V = o(r^2)$ as $r \to 0$, a positive lower bound estimate for the distance between particles.*

7.3.2 In Search of a Nontrivial Integral of Motion

It is not difficult to see that the center of the Poisson algebra $\mathcal{P}_I(Q_{\mathbf{S}})$, commuting with the two-body Hamiltonian function, is not wide enough to imply the integrability of the two-body problem with a nontrivial potential. Now it is not known any central potential admitting a nontrivial integral of motion, i.e., an integral depending on both the variables r, p_r and the group variables $p_0, p_1, \ldots$ (and certainly independent from the Hamiltonian function). In Sect. 7.5 we shall see that the existence of one such integral for the real case would lead to the integrability of the two-body problem. Here we shall describe one possible approach to the search of such integral.

Obviously, the two-body problem with the trivial potential $V \equiv 0$ is integrable, since in this case particles move independently. Similarly to the quantum case (see Remark 5.3), the two-body Hamiltonian function can be represented in the form $h_1 + h_2 + V(r)$, where the function h_k, $k = 1, 2$ is proportional to $1/m_k$ and it holds $[h_1, h_2]_P = 0$. For example, for the function (7.3) one has:

$$\begin{aligned}
h_1 &= \frac{(1+r^2)^2}{8m_1R^2}p_r^2 - \frac{\beta(1+r^2)}{2m_1R^2}p_rp_0 + \frac{\beta^2}{2m_1R^2}p_0^2 \\
&+ \frac{1+r^2}{2m_1R^2r^2}\left(\sin^2(\beta\arctan r)p_1 + \cos^2(\beta\arctan r)p_2 - \sin(2\beta\arctan r)p_3\right) \\
&+ \frac{(1+r^2)^2}{8m_1R^2r^2}\left(\sin^2(2\beta\arctan r)p_4 + \cos^2(2\beta\arctan r)p_5 - \sin(4\beta\arctan r)p_6\right), \\
h_2 &= \frac{(1+r^2)^2}{8m_2R^2}p_r^2 + \frac{\alpha(1+r^2)}{2m_2R^2}p_rp_0 + \frac{\alpha^2}{2m_2R^2}p_0^2 \\
&+ \frac{1+r^2}{2m_2R^2r^2}\left(\sin^2(\alpha\arctan r)p_1 + \cos^2(\alpha\arctan r)p_2 + \sin(2\alpha\arctan r)p_3\right) \\
&+ \frac{(1+r^2)^2}{8m_2R^2r^2}\left(\sin^2(2\alpha\arctan r)p_4 + \cos^2(2\alpha\arctan r)p_5 + \sin(4\alpha\arctan r)p_6\right).
\end{aligned}$$

Under the choice $\alpha = 1, \beta = 0$ functions h_1 and h_2 are rational w.r.t. the variable r:

$$h_1 = \frac{(1+r^2)^2}{8m_1R^2}p_r^2 + \frac{1+r^2}{2m_1R^2r^2}p_2 + \frac{(1+r^2)^2}{8m_1R^2r^2}p_5,$$
$$h_2 = \frac{(1+r^2)^2}{8m_2R^2}p_r^2 + \frac{1+r^2}{2m_2R^2}p_rp_0 + \frac{p_0^2}{2m_2R^2} + \frac{r^2p_1+p_2+2rp_3}{2m_2R^2r^2}$$
$$+ \frac{1}{8m_2R^2r^2}\left(4r^2p_4 + (1-r^2)^2p_5 + 4r(1-r^2)p_6\right) .$$

Nevertheless it is not clear how to incorporate a nontrivial potential into functions h_1, h_2 not disturbing their commutativity and in such a way that it would be $h = h_1 + h_2$.

Numerical calculations of the Poincaré surfaces of sections by the author and I. E. Stepanova for reduced two-body systems on spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$ (see Sect. 7.5) with Coulomb and oscillatory potentials discovered a soft chaos in these dynamical systems [161]. The meromorphic nonintegrability of these systems for $n = 2$ was proved in [171].

7.4 The Center of Mass Problem on Two-Point Homogeneous Spaces

The importance of the mass center for an isolated system of particles or a rigid body in Euclidean space stems from its following properties:

1. the mass center of a classical mechanical system moves with a constant speed along a (geodesic) line;
2. variables corresponding to the mass center are separated from other variables both in classical and quantum mechanical problems.

These properties imply, in particular, that the (generally complicated) motion of a classical system can be decomposed into the motion of the center of mass and the motion of the system with respect to this center, often greatly simplifying the problem. Under the action of external forces the center of mass moves as if all forces act on the particle located at the center of mass and having the mass equal to the total mass of the system. An attempt to generalize the concept of the center of mass for the curved two-point homogeneous Riemannian spaces encounters difficulties related to the absence of nice dynamical properties such as 1 and 2 above. It is natural to define the mass center for the two particles on a two-point homogeneous Riemannian space as the point on the shortest geodesic interval joining these particles that divides the interval in a fixed ratio. If this ratio is equal to the ratio of particle masses (as for the center mass concept in Euclidean space), we denote the corresponding mass center by R_1.

However, even for spaces of constant sectional curvature, such a mass center does not have property 1 [160]. For example, consider two free particles on the sphere $\mathbf{S}^2$. Choose two antipodal points on the sphere (poles) and the equator corresponding to these poles. Let one particle rests at one pole and another moves with the constant speed along the equator. Then any point on the shortest geodesic interval joining such particles does not move along a geodesic unless this point coincides with one of the particles. The latter is obviously senseless. Therefore, for a definition of the mass center on a two-point homogeneous Riemannian space we must rely on properties different from the property 1.

7.4.1 Existing Mass Center Concepts for Spaces of a Constant Curvature

The axiomatic approach to the concept of mass center for spaces of a constant curvature was developed in [47, 48]. Let $\mathfrak{A} = ((A_i, m_i))_{i=1}^N$ be a system (possibly empty) of mass points A_i with masses m_i in the space Q of a constant sectional curvature, which corresponds to the types 2 or 9 according to the classification given in Sect. 1.1. Denote by $\mathcal{A}$ the set of all such systems and by $\mathcal{A}_0$ the subset of one-particle systems. For any positive real number χ define the operation $\chi \cdot \mathfrak{A} = ((A_i, \chi m_i))_{i=1}^N$.

Theorem 7.2 ([48]). *Let Q be a space of a constant curvature $\mathbf{S}^n$ or $\mathbf{H}^n(\mathbb{R})$. There is a unique map $\mathbb{U}$ of the set $\mathcal{A}$ onto the set $\mathcal{A}_0$, satisfying the following axioms:*
1) $\mathbb{U}((A_1, m_1)) = (A_1, m_1)$;
2) $\mathbb{U}(\mathfrak{A} \cup \mathfrak{B}) = \mathbb{U}(\mathbb{U}(\mathfrak{A}) \cup \mathbb{U}(\mathfrak{B}))$;
3) $\mathbb{U}(\chi \cdot \mathfrak{A}) = \chi \cdot \mathbb{U}(\mathfrak{A})$;
4) $\mathbb{U} \circ q = q \circ \mathbb{U}$, where q is an arbitrary isometry of the space Q;
5) the map $\mathbb{U}$ is continuous with respect to the natural topology on the space $\mathcal{A}$. Two systems are close to each other in this topology, if their mass points are pairwise close and have similar masses. Points with small masses are close to the empty set.

For the sphere $\mathbf{S}^n$ this map $\mathbb{U}$ moves the system $((A_1, m_1), (A_2, m_2))$ into the mass point (mass center), located on the geodesic interval joining the points A_1, A_2. The distances ρ_i, $i = 1, 2$ between the mass center and the points A_i are determined from the following equations:

$$m_1 \sin\left(\frac{\rho_1}{R}\right) = m_2 \sin\left(\frac{\rho_2}{R}\right), \ \rho_1 + \rho_2 = \rho,$$

where ρ is the distance between particles. The mass of the mass center is assumed to be

$$m_1 \cos\left(\frac{\rho_1}{R}\right) + m_2 \cos\left(\frac{\rho_2}{R}\right) .$$

For the Lobachevski space $\mathbf{H}^n(\mathbb{R})$ the map $\mathbb{U}$ is obtained by using the hyperbolic functions sinh, cosh *instead of the corresponding trigonometric functions* sin, cos. *For example for two mass points one has:*

$$m_1 \sinh\left(\frac{\rho_1}{R}\right) = m_2 \sinh\left(\frac{\rho_2}{R}\right), \ \rho_1 + \rho_2 = \rho , \tag{7.31}$$

and the mass of the mass center is

$$m_1 \cosh\left(\frac{\rho_1}{R}\right) + m_2 \cosh\left(\frac{\rho_2}{R}\right) .$$

We denote the mass center defined in this way by R_2. Note that this mass center for two particles with equal masses located at the diametrically opposite points of a sphere has an arbitrary position on the corresponding equator and the null mass, which is equivalent to the empty set.

This approach to the definition of the center of mass corresponds to the mass center concept in the flat space-time of special relativity (SR) [48]. In fact, for a given inertial frame of reference, there exists a one-to-one correspondence between possible particles' velocities in SR and mass points in the space $\mathbf{H}^3(\mathbb{R})$, with masses equal to the rest masses in SR. Therefore, a system $\mathfrak{A} = ((A_i, m_i))_{i=1}^N \in \mathcal{A}$ corresponds to a system $\varsigma(\mathfrak{A})$ of moving particles in SR with rest masses m_i and velocities $\mathbf{v}_i$. The total mass m and momentum $\mathbf{p}$ of the latter system are defined by the following equalities

$$m := \sum_{i=1}^{N} \frac{m_i}{\sqrt{1-\mathbf{v}_i^2}}, \ \mathbf{p} := \sum_{i=1}^{N} \frac{m_i \mathbf{v}_i}{\sqrt{1-\mathbf{v}_i^2}} \ .$$

One can define an effective particle Ξ in SR with the rest mass m and the velocity $\mathbf{v}$ such that

$$\frac{m\mathbf{v}}{\sqrt{1-\mathbf{v}^2}} = \mathbf{p} \ .$$

This particle determines the mass center $\varsigma^{-1}(\Xi)$ of the system $\mathfrak{A}$ in the space $\mathbf{H}^3(\mathbb{R})$, which geometrically coincides with R_2.

It is clear that this definition of a mass center can be easily generalized to systems with a distributed mass.

The definition of the mass center R_2 seems to be quite natural. Unfortunately, no "good" dynamical properties are known for it. In order to find the mass center with such properties, one can try to search for a pure geometrical mass center without any mass. In this case one need not be concerned about the validity of axioms 2 and 3 of Theorem 7.2 and thus has more freedom. This approach to the mass center concept concerning the free motion on spaces $\mathbf{S}^n$, $\mathbf{H}^n(\mathbb{R})$, $n = 2, 3$ was developed with various degrees of generality in [128, 152, 215]. Consider the following definition of a mass center, used in these papers. Let $Q = \mathbf{H}^n(\mathbb{R})$, $n = 2, 3$. Define a rigid body in Q by a non-negative density function $\varrho(x)$, $x \in Q$ with a compact connected support and consider the function

$$\Upsilon(x) = \int_Q \sinh^2\left(\frac{\rho(x,y)}{R}\right) \varrho(y) d\mu(y) \ , \tag{7.32}$$

where μ is the measure on the space Q, generated by the Riemannian metric. This function has a unique minimum and the coordinate of this minimum can be chosen as a definition of the center of mass R_3 for the rigid body. It is clear that the similar definition can be given for a system of particles, replacing the integral in (7.32) by the corresponding sum.

Unlike the center of mass R_2, the mass center R_3 for two particles is determined from the following equation (cf. (7.31))

$$m_1 \sinh\left(\frac{2\rho_1}{R}\right) = m_2 \sinh\left(\frac{2\rho_2}{R}\right) , \ \rho_1 + \rho_2 = \rho \ . \tag{7.33}$$

Here as before ρ is the distance between the particles and ρ_i, $i = 1, 2$ is the distance between the ith particle and the mass center located on the shortest geodesic interval joining the particles.

Call a free movement of a rigid body a *free rotation* if all points of this body move along trajectories of some elliptic transformation (see Sect. 1.3.3). Call a free movement of a rigid body a *free transvection* if all points of this body move along trajectories of some transvection. The mass center R_3 has the following dynamical properties.

1. A free rotation of a rigid body about its mass center is possible in the space $\mathbf{H}^n(\mathbb{R})$. For $n = 2$ there is only one such rotation [128] and for $n = 3$ there are three independent rotations [152] about three pairwise perpendicular axes passing through the mass center R_3.
2. All possible free transvections of a rigid body have axes passing through the mass center R_3. For $n = 2$ there are two such geodesics. For $n = 3$ there are three such geodesics and they coincide with the axes of free rotations.
3. The mass center R_3 is uniquely determined by any of the properties 1 or 2.
4. The velocities of all possible free rotations and transvections are constant.

Note that there are no free movements of a rigid body along horocycles [128].

The situation for the spaces $Q = \mathbf{S}^n$, $n = 2, 3$ is analogous if we restrict ourselves to rigid bodies of "moderate" sizes, i.e., if the diameter of a rigid body is no more than $\pi R/4$ [215]. This condition is required in order to differ transvections and rotations of rigid bodies by the location of immovable points of corresponding one-parameter isometry subgroups with respect to the rigid body itself, since all such subgroups of the isometry group $\mathrm{SO}(n+1)$ are conjugated in $\mathrm{SO}(n+1)$, and their trajectories in the space Q are equivalent.

Note that most of free movements of a rigid body in constant curvature spaces do not correspond to the center of mass R_3 movement along a geodesic even when this rigid body is a homogeneous ball [215]. Due to this fact and property 1 above there are no points in a rigid body that move along geodesics for an arbitrary initial velocities. Since a rigid body is a limiting case of a system of interactive particles, the same is also valid for such systems.

For a system of two particles we shall try to find a point on the shortest geodesic interval, joining particles, that divides it in a definite ration such that this point moves with a constant speed along a geodesic *for some initial particles' velocities and an arbitrary interactive potential.*

Definition 7.1. *Let $Q_2 = (Q \times Q)\backslash Q_{op}$ be a set of two-particle positions that correspond to the only one shortest path joining particles. A map from Q_2 to Q is called the dynamical mass center if*

1. *it maps a two-particle position from Q_2 to the point $\mathbf{x}_0$ on the geodesic interval joining the particles that divides the length of this interval in some ratio depending only on particles' masses;*
2. *for any geodesic on Q there should be some initial positions and velocities of particles such that for any interactive potential the point $\mathbf{x}_0$ moves along this geodesic with a constant speed.*

For brevity, we call the value of this map the "dynamical mass center".

Note that this definition is appropriate for any complete Riemannian space if particles are sufficiently close to each other.

7.4.2 The Connection of Existing Mass Center Concepts with the Two-Body Hamiltonian Functions

Consider now the connection between formulas for the two-body Hamiltonian functions from Sects. 7.1, 7.2 and different mass center concepts.

Suppose that the geodesic interval $\widetilde{\gamma}(s)$ from Sect. 5.1, joining particles, moves simultaneously with their motion. The point $\widetilde{\gamma}(0)$ divides the geodesic interval in the ratio $\alpha/(1-\alpha)$, $0 \leqslant \alpha \leqslant 1$. On the other hand, the expansion of the Lie algebra $\mathfrak{g}$ in Proposition 1.2 is specialized for the point $\widetilde{\gamma}(0)$. Denote by $w(t)$ the evolution of the point $\widetilde{\gamma}(0)$. This function can be discontinuous, when particles are in $Q_{op} \subset Q \times Q$, but for $Q \neq \mathbf{P}^n(\mathbb{R})$ a general trajectory on $Q \times Q$ does not intersect Q_{op}.

Consider the simplest compact case $Q = \mathbf{S}^2$ or $Q = \mathbf{P}^2(\mathbb{R})$ with the Hamiltonian function given by (7.13). Here the group K_0 is trivial and $(Q \times Q)\backslash(\mathrm{diag} \cup Q_{op}) = I \times \mathrm{SO}(3)$. Therefore the function $-p_0$ is the Hamiltonian function of the vector field $\widetilde{\Lambda}^l$ on $T^*\,\mathrm{SO}(3)$ (see (4.27), (4.28) and the proof of Proposition 4.9). Here we identifies the initial position of the system on $(Q \times Q)\backslash(\mathrm{diag} \cup Q_{op}) = I \times \mathrm{SO}(3)$ with the point $(r(0), e) \in I \times \mathrm{SO}(3)$.

Clearly, the projection of the system motion onto the second factor of the product $I \times \mathrm{SO}(3)$ is described by the map:

$$t \to \exp(tc\Lambda)$$

if the projection $\widetilde{dh}$ of the differential dh onto the second factor of the product $T^*I \times T^*\,\mathrm{SO}(3)$ equals $-cdp_0$, where $c = \mathrm{const}$ along the trajectory of the system. This case corresponds to a particle motion along a common geodesic in such a way that the motion of the point w on Q is described by the formula

$$w(t) = \exp(tc\Lambda)w(0)$$

and due to the second claim of Proposition 5.1 the point w moves along the geodesic $\widetilde{\gamma}|_{t=0}$ with a constant speed.

It remains to verify the condition $\widetilde{dh} = -cdp_0$ on the trajectory of the system. Evidently

$$\begin{aligned}\widetilde{dh} = &\left(\frac{(m_1\alpha - m_2\beta)(1+r^2)}{2m_1m_2R^2}p_r + \frac{m_1\alpha^2 + m_2\beta^2}{m_1m_2R^2}p_0\right)dp_0 \\ &+ (C_sp_1 + B_sp_2)\,dp_1 + (A_sp_2 + B_sp_1)\,dp_2\;.\end{aligned} \tag{7.34}$$

For an arbitrary potential V the values r and p_r can be arbitrary, therefore $\widetilde{dh} = -cdp_0$ iff $m_1\alpha = m_2\beta$, $p_0 = \mathrm{const}$, $p_1 \equiv p_2 \equiv 0$. But due to (7.14) these values of p_i are really conserved along a trajectory of the system. Thus, one gets:

Proposition 7.1. *For two particles on a two-point homogeneous Riemannian space the center of mass R_1 is the only one dynamical mass center corresponding to a particles motion along a common geodesic.*

Proof. For the space $Q = \mathbf{S}^2$ or $Q = \mathbf{P}^2(\mathbb{R})$ the claim follows from the considerations above, since under the condition $m_1\alpha = m_2\beta$ the point $\widetilde{\gamma}(0)$ coincides with the mass center R_1. For other compact two-point homogeneous spaces an every geodesic lies on a two-dimensional completely geodesic submanifold (see Remark 1.1) that reduces the consideration to the case $Q = \mathbf{S}^2, \mathbf{P}^2(\mathbb{R})$. For noncompact two-point homogeneous spaces the consideration is similar due to Remark 5.2. □

Consider now the possibility of the motion of the same system, corresponding to Hamiltonian function (7.13) under the action of the one-parametric group, generated by the element $f_{2\lambda,1} \in \mathfrak{so}(3)$. This motion is generated by the differential cdp_2, $c \neq 0$ and corresponds to the rotation of the geodesic $\widetilde{\gamma}$ about the fixed point $\widetilde{\gamma}(0)$. Arguing as above one concludes that in this case the expression for $\widetilde{dh}$ should be cdp_2. From the expression (7.34) one gets:

$$\begin{gathered} C_s p_1 + B_s p_2 = 0,\ B_s p_1 + A_s p_2 = c, \\ \frac{(m_1\alpha - m_2\beta)(1+r^2)}{2m_1 m_2 R^2} p_r + \frac{m_1\alpha^2 + m_2\beta^2}{m_1 m_2 R^2} p_0 = 0\,. \end{gathered} \tag{7.35}$$

As above for a general potential $V(r)$ values of r and p_r are arbitrary and the last equation from (7.35) gives again $\alpha = m_2/(m_1+m_2)$ and then $p_0 \equiv 0$. For the same reasons the first equation implies $p_1 \equiv 0$, $p_2 \equiv 0$ and thus $c = 0$. Therefore, the property 1 of the mass center R_3 of a rigid body is not valid for two particles and an arbitrary potential.

Suppose now that two particles are joined along the geodesic $\widetilde{\gamma}$ by the weightless rod. This system (barbell) is a particular case of a rigid body. Consider the possibility of the same rotation for this system. The phase space for it is $T^*\,\mathrm{SO}(3)$ and the Hamiltonian function can be obtained from (7.13) by setting $r = \mathrm{const}, p_r \equiv 0$, $V(r) = \mathrm{const}$, (for simplicity $V(r) \equiv 0$). The last equation from (7.35) now gives $p_0 \equiv 0$ and therefore

$$\frac{dp_0}{dt} = (A_s - C_s)p_1 p_2 + B_s(p_1^2 - p_2^2) = 0$$

due to (7.14) and (7.26).

If $B_s(r) \neq 0$, then from the first equation (7.35) one gets $p_2 = -C_s p_1/B_s$ and

$$\frac{dp_0}{dt} = \frac{B_s^2 - A_s C_s}{B_s} p_1^2 = 0\,.$$

Due to the definition of A_s, B_s and C_s in Sect. 5.2 it holds $A_s C_s - B_s^2 \neq 0$, therefore $p_1 \equiv 0$, $p_2 \equiv 0$ and there is no any rotation.

The possibility $B_s = 0$ implies $p_1 \equiv 0$, $p_2 \equiv c/A_s \neq 0$. Again due to (7.14) and (7.26), these values are conserved along the trajectory of the system.

This means that the pure rotation of the barbell is possible about the fixed point $\widetilde{\gamma}(0)$, defined by the equation:

$$m_1 \sin\left(4\alpha \arctan r\right) = m_2 \sin\left(4(1-\alpha) \arctan r\right) \ . \tag{7.36}$$

Since the distances ρ_i, $i = 1, 2$ between the ith particle and the point $\widetilde{\gamma}(0)$ are $\rho_1 = 2R\alpha \arctan r$, $\rho_2 = 2R(1-\alpha) \arctan r$, equation (7.36) is the analogue of (7.33) that corresponds to the property 1 of the mass center R_3 of a rigid body.

Reasoning in the similar way one obtains the following. The transvection of the point w along the trajectory of the one parameter group $\exp(te_{\lambda,1})$ is realized iff $\widetilde{dh} = cdp_1$, where $c = \mathrm{const}$ along a trajectory of the system. For fixed initial conditions and an arbitrary potential this is impossible for the system of two particles. Thus the property 2 of the mass center R_3 of a rigid body is not valid for two particles and an arbitrary potential. For the barbell the condition $\widetilde{dh} = cdp_1$ along a trajectory is realized iff $p_0 \equiv 0$, $p_2 \equiv 0$, $p_1 = c/C_s \neq 0$, $B_s = 0$ that in particular implies (7.36). This is the realization of the property 2 of the mass center R_3 for the barbell.

The similar rotation and transvection are possible also for a general compact and noncompact two-point homogeneous spaces on a completely geodesic subspaces of a constant curvature.

7.5 Hamiltonian Reduction of the Two-Body Problem on Constant Curvature Spaces

Here we consider the Hamiltonian reduction of the two-body problem on the spaces $\mathbf{S}^n$ and $\mathbf{H}^n(\mathbb{R})$, using results of Sects. 7.1.4, 7.2.4 and Theorem 4.6. This reduction was carried out by pure coordinate evaluation in [160, 166] and in invariant form in [163, 165].

7.5.1 Hamiltonian Reduction of the Two-Body Problem on Spheres

As was mentioned above in Sect. 7.3 the classical two-body problem on the spheres $\mathbf{S}^n$ reaches its full generality at $n = 3$.

If $n = 1$ the system posses two-degrees of freedom and one integral associated with the action of the symmetry group $\mathrm{SO}(2) \cong \mathbf{S}^1$. Therefore, it is integrable for every potential $V(r)$. Similar to (7.10), (7.13) the expression for the Hamiltonian function in this case for $\alpha = m_2/(m_1 + m_2)$ looks like

$$h = \frac{(1+r^2)^2}{8mR^2} p_r^2 + \frac{p_0^2}{2(m_1+m_2)R^2} + V(r) \ ,$$

where $p_0 = \mathrm{const}$. Evidently, the reduced phase space is $T^*(\mathbf{S}^1 \setminus \mathrm{pt}) \cong T^*\mathbb{R}$ with canonical coordinates r, $0 < r < \infty$ and p_r.

Suppose now $n = 2$. In this case one has expansion (7.2) of the phase space

$$\left(T^*I \times T^*\mathbf{S}^2_S\right) \cup \widetilde{T}^*Q_{op} =: M_{ess} \cup \widetilde{T}^*Q_{op},$$

where $Q_{op} = \mathbf{S}^2$. Here $\dim M_{ess} = 8$, $\dim \widetilde{T}^*Q_{op} = 6$ and a typical trajectory of the system does not intersects $\widetilde{T}^*Q_{op}$. One can reduce the space $T^*\mathbf{S}^2_S$ w.r.t.

the SO(3)-action using Theorem 4.6. Since in this case the group K_0 is trivial, the space $T^*\mathbf{S}^2_S$ is reduced to the $\mathrm{Ad}^*_{\mathrm{SO}(3)}$-orbit (see Remark 4.6) defined by the equation

$$p_0^2 + p_1^2 + p_2^2 = c = \text{const} \geqslant 0 \ ,$$

where p_0, p_1, p_2 are linear coordinates on $\mathfrak{so}^*(3)$. Thus, the reduced space for M_{ess} and $c > 0$ is

$$\widetilde{M}_{ess,c} = T^*I \times \mathbf{S}^2 \ ,$$

where the symplectic structure on $\mathbf{S}^2 \subset \mathfrak{so}^*(3) \cong \mathbb{E}^3$ is the area form up to a multiplicative constant and the reduced Hamiltonian function for $\alpha = m_2/(m_1+m_2)$ is

$$\tilde{h} = \frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{p_0^2}{2(m_1+m_2)R^2} + \frac{1}{2}\left(C_s(r)p_1^2 + A_s(r)p_2^2 + 2B_s(r)p_1p_2\right) + V(r) \ .$$

A suitable pair of functions from the triple p_0, p_1, p_3 is local coordinates on $\mathbf{S}^2$ and their Poisson brackets are defined from (7.14).

Evidently, there is a two-dimensional invariant submanifold in $\widetilde{M}_{ess,c}$ defined by the equations $p_1 \equiv p_2 \equiv 0$. It corresponds to the bodies motion along a common geodesic.

Moreover, for $m_1 = m_2 = 2m$ one has

$$\tilde{h} = \frac{1}{8mR^2}\left((1+r^2)^2p_r^2 + p_0^2 + (1+r^2)\left(p_1^2 + \frac{p_2^2}{r^2}\right)\right) + V(r)$$

that implies due to (7.14) and (7.26) the existence of two other invariant submanifolds. The first one is defined by equations $p_0 \equiv p_2 \equiv 0$ and the second one by equations $p_0 \equiv p_1 \equiv 0$. These manifolds correspond to a pure transvection and a pure rotation of the system (cf. Sect. 7.4.2), but generally with nonconstant velocity.

The value $c = 0$ corresponds to the integrable system with one degree of freedom. Here $\widetilde{M}_{ess,0} = T^*I,\ p_0 \equiv p_1 \equiv p_2 \equiv 0$ and

$$\tilde{h} = \frac{(1+r^2)^2}{8mR^2}p_r^2 + V(r) \ . \tag{7.37}$$

In the case $n = 3$ the situation is more subtle. Now $M_{ess} := T^*I \times T^*(\mathrm{SO}(4)/\mathrm{SO}(2))$ and one should reduce the space $T^*(\mathrm{SO}(4)/\mathrm{SO}(2))$. The Lie algebra $\mathfrak{so}(4)$ has the base (see Sect. 3.4.1) Ψ_{ij}, $1 \leqslant i < j \leqslant 4$. Let ψ_{ij}, $1 \leqslant i < j \leqslant 4$ be coordinates in $\mathfrak{so}^*(4)$, corresponding to the base Ψ^{ij}, $1 \leqslant i < j \leqslant 4$, dual to the base Ψ_{ij}, $1 \leqslant i < j \leqslant 4$. Let

$$\begin{aligned} &\tilde{e}_1 = -\Psi_{23} - \Psi_{14},\ \tilde{e}_2 = -\Psi_{24} + \Psi_{13},\ \tilde{e}_3 = -\Psi_{12} - \Psi_{34}, \\ &\tilde{e}_4 = \Psi_{23} - \Psi_{14},\ \tilde{e}_5 = \Psi_{24} + \Psi_{13},\ \tilde{e}_6 = \Psi_{12} - \Psi_{34} \end{aligned} \tag{7.38}$$

be another base in $\mathfrak{so}(4)$. The commutator relations for $\tilde{e}_i$, $i = 1, \ldots, 6$ are (see (1.12)):

$$\begin{aligned} &[\tilde{e}_1, \tilde{e}_2] = \tilde{e}_3,\ [\tilde{e}_2, \tilde{e}_3] = \tilde{e}_1,\ [\tilde{e}_3, \tilde{e}_1] = \tilde{e}_2,\ [\tilde{e}_4, \tilde{e}_5] = \tilde{e}_6,\ [\tilde{e}_5, \tilde{e}_6] = \tilde{e}_4, \\ &[\tilde{e}_6, \tilde{e}_4] = \tilde{e}_5,\ [\tilde{e}_i, \tilde{e}_j] = 0,\ 1 \leqslant i \leqslant 3,\ 4 \leqslant j \leqslant 6 \ . \end{aligned}$$

This means that the triples $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ and $\tilde{e}_4, \tilde{e}_5, \tilde{e}_6$ correspond to the expansion $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and the base $\tilde{e}_i$, $i = 1, \ldots, 6$ is orthonormal w.r.t. the Killing form on $\mathfrak{so}(4)$.

Let $\tilde{p}_i$, $i = 1, \ldots, 6$ be coordinates in $\mathfrak{so}^*(4)$, corresponding to the base $\tilde{e}^i$, $i = 1, \ldots, 6$, dual to $\tilde{e}_i$, $i = 1, \ldots, 6$. An $\mathrm{Ad}^*_{\mathrm{SO}(4)}$-orbit $\mathcal{O}_\beta$ is defined by equations

$$I_1 := \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 = \beta_1^2,\ I_2 := \tilde{p}_4^2 + \tilde{p}_5^2 + \tilde{p}_6^2 = \beta_2^2;\ \beta_1, \beta_2 \in \mathbb{R};\ \beta_1, \beta_2 \geqslant 0$$

or equivalently by equations

$$\begin{aligned} I_1 &= (\psi_{23} + \psi_{14})^2 + (\psi_{13} - \psi_{24})^2 + (\psi_{12} + \psi_{34})^2 = \beta_1^2, \\ I_2 &= (\psi_{23} - \psi_{14})^2 + (\psi_{13} + \psi_{24})^2 + (\psi_{12} - \psi_{34})^2 = \beta_2^2\ . \end{aligned} \tag{7.39}$$

In the general case $\beta_1 \neq 0, \beta_2 \neq 0$ these orbits are diffeomorphic to $\mathbf{S}^2 \times \mathbf{S}^2$.

Another equivalent form of $\mathrm{Ad}^*_{\mathrm{SO}(4)}$-invariants is

$$\begin{aligned} J_1 &:= \frac{1}{2}(I_1 + I_2) = \psi_{12}^2 + \psi_{13}^2 + \psi_{14}^2 + \psi_{23}^2 + \psi_{24}^2 + \psi_{34}^2\ , \\ J_2 &:= \frac{1}{4}(I_1 - I_2) = \psi_{23}\psi_{14} - \psi_{13}\psi_{24} + \psi_{12}\psi_{34}\ . \end{aligned} \tag{7.40}$$

In notations of Sect. 3.4.1 the stationary Lie subalgebra $\mathfrak{k}_0 \cong \mathfrak{so}(2)$, corresponding to a general two-body position in $\mathbf{S}^3$, is generated by the element Ψ_{34}. Therefore, the subspace $\mathrm{ann}\,\mathfrak{k}_0 \subset \mathfrak{so}^*(4)$ is defined by the equation $\psi_{34} = 0$ or equivalently by $\tilde{p}_3 + \tilde{p}_6 = 0$.

Due to definition of operators $D_1, D_2, \Box$ in Sect. 3.4.1 and their correspondence with functions $p_1, p_2, p_\Box$ from Sect. 7.1.4 one has

$$\begin{aligned} &p_0 = -2\psi_{12},\ p_1 = 4(\psi_{13}^2 + \psi_{14}^2),\ p_2 = 4(\psi_{23}^2 + \psi_{24}^2)\ , \\ &p_3 = -4\left(\psi_{13}\psi_{23} + \psi_{14}\psi_{24}\right),\ p_\Box = 4\left(\psi_{13}\psi_{24} - \psi_{14}\psi_{23}\right) \end{aligned} \tag{7.41}$$

on the subspace $\mathrm{ann}\,\mathfrak{k}_0$ and therefore

$$p_0^2 + p_1 + p_2 = 2(\beta_1^2 + \beta_2^2),\ p_\Box = \beta_2^2 - \beta_1^2\ .$$

Let us verify Assumption 4.3, which implies Assumption 4.2. Consider a point $z \in \mathcal{O}'_\beta := \mathcal{O}_\beta \cap \mathrm{ann}\,\mathfrak{k}_0$ with coordinates $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5, \tilde{p}_6 = -\tilde{p}_3)$. First suppose $\beta_1 \neq 0, \beta_2 \neq 0$. Then a vector $Y = \sum_{i=1}^6 y_i \tilde{e}^i \in T_z\mathfrak{so}^*(4)$ belongs to $T_z\mathcal{O}_\beta$ iff

$$\tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 = 0,\ \tilde{p}_4 y_4 + \tilde{p}_5 y_5 - \tilde{p}_3 y_6 = 0\ . \tag{7.42}$$

Since $\dim \mathrm{ann}\,\mathfrak{k}_0 = 5$, the orbit $\mathcal{O}_\beta$ is not transversal to $\mathrm{ann}\,\mathfrak{k}_0$ at the point z iff $T_z\mathcal{O}_\beta \subset \mathrm{ann}\,\mathfrak{k}_0$. Via coordinates the last inclusion means that the system (7.42) implies $y_3 + y_6 = 0$. Since equations in (7.42) are independent from each other, it is possible only if $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}_4 = \tilde{p}_5 = 0$, $\tilde{p}_3 \neq 0$ and then $\beta_1 = \beta_2$.

Let now $\beta_1 \neq 0, \beta_2 = 0$ and then $\tilde{p}_3 = \tilde{p}_4 = \tilde{p}_5 = \tilde{p}_6 = 0$. In this case a vector $Y = y_1\tilde{e}^1 + y_2\tilde{e}^2 + y_3\tilde{e}^3$ belongs to $T_z\mathcal{O}_\beta$ iff

$$\tilde{p}_1 y_1 + \tilde{p}_2 y_2 = 0 \,. \tag{7.43}$$

Since y_3 does not occur in (7.43), the condition $y_3 = 0$ is not a corollary of (7.43) and $\mathcal{O}_\beta$ is transversal to $\operatorname{ann} \mathfrak{k}_0$ at the point z. The same is also valid for $\beta_1 = 0, \beta_2 \neq 0$.

Thus, for orbits $\mathcal{O}_\beta$ with $\beta_1 \neq \beta_2$, $\beta_1 \geqslant 0, \beta_2 \geqslant 0$ Assumptions 4.3 and 4.2 are satisfied. Let $\mathcal{O}_\beta$ be such an orbit. Then $\mathcal{O}'_\beta = \mathcal{O}_\beta \cap \operatorname{ann} \mathfrak{k}_0$ is a smooth submanifold in $\mathfrak{so}^*(4)$. Since

$$[\Psi_{34}, \tilde{e}_1] = -\frac{1}{2}\tilde{e}_2,\ [\Psi_{34}, \tilde{e}_2] = \frac{1}{2}\tilde{e}_1,\ [\Psi_{34}, \tilde{e}_4] = -\frac{1}{2}\tilde{e}_5,\ [\Psi_{34}, \tilde{e}_5] = \frac{1}{2}\tilde{e}_4,$$

the $\mathrm{Ad}^*_{K_0}$-action is the synchronous orthogonal rotation in 2-planes in the space $\mathfrak{so}^*(4)$ generated by pairs $\tilde{e}^1, \tilde{e}^2$ and $\tilde{e}^4, \tilde{e}^5$, respectively. Due to $\beta_1 \neq \beta_2$ the submanifold $\mathcal{O}'_\beta$ does not contain points with coordinates $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}_4 = \tilde{p}_5 = 0$ and the $\mathrm{Ad}^*_{K_0}$-action on $\mathcal{O}'_\beta$ is free that implies the Assumption 4.4. Thus, the factor space $\widetilde{\mathcal{O}}_\beta := \mathcal{O}'_\beta / \mathrm{Ad}^*_{K_0}$ is a smooth manifold.

1. Suppose $\beta_1 > 0$, $\beta_2 > 0$, $\beta_1 \neq \beta_2$. In this case $\widetilde{\mathcal{O}}_\beta$ is diffeomorphic to the sphere $\mathbf{S}^2$. Indeed, if $\beta_1 > \beta_2$, then any $\mathrm{Ad}^*_{K_0}$-orbit in $\mathcal{O}'_\beta$ contains a unique point with coordinates $\tilde{p}_1 = \sqrt{\beta_1^2 - \tilde{p}_3^2} > 0, \tilde{p}_2 = 0, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5, \tilde{p}_6 = -\tilde{p}_3$ which satisfy the equation

$$\tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_5^2 = \beta_2^2 \,. \tag{7.44}$$

This means that $\mathrm{Ad}^*_{K_0}$-orbits in $\mathcal{O}'_\beta$ are in one-to-one correspondence with points of the sphere $\mathbf{S}^2 \subset \mathfrak{so}^*(3) \cong \mathbb{E}^3$, defined by (7.44), and any local coordinates on $\mathbf{S}^2$ correspond to local coordinates on $\widetilde{\mathcal{O}}_\beta$. In the case $\beta_2 > \beta_1 > 0$ the consideration is similar.
Thus, due to Theorem 4.6, in the case $\beta_1, \beta_2 > 0$, $\beta_1 \neq \beta_2$ the reduced phase space for M_{ess} is $\widetilde{M}_{ess,\beta} = T^*I \times \mathbf{S}^2$ and the reduced Hamiltonian function is defined by (7.10), where

$$p_0^2 + p_1 + p_2 = 2(\beta_1^2 + \beta_2^2),\ p_1 p_2 - p_3^2 = p_\square^2 = (\beta_1^2 - \beta_2^2)^2 \,.$$

Any two functions from the quadruple p_0, p_1, p_2, p_3 are independent on $\widetilde{M}_{ess,\beta}$. The commutator relations for these functions are in (7.11). The reduced Hamiltonian system has two degrees of freedom and for its integrability one needs an additional integral, independent on the Hamiltonian function. Such integral is not known for any nontrivial potential.
2. Consider now the case $\beta_1 = 0, \beta_2 > 0$. Here the submanifold $\mathcal{O}'_\beta$ is the circle defined by the equations

$$\tilde{p}_1 = \tilde{p}_2 = \tilde{p}_3 = \tilde{p}_6 = 0,\ \tilde{p}_4^2 + \tilde{p}_5^2 = \beta_2^2$$

and the group K_0 acts on it freely and transitively. Thus, the reduced space for M_{ess} is $\widetilde{M}_{ess,\beta} = T^*I$. Moreover it holds $\psi_{12} = 0, \psi_{23} = -\psi_{14}, \psi_{24} = \psi_{13}$ on $\mathcal{O}'_\beta$ and therefore due to (7.41) $p_0 = p_3 = 0$, $p_1 = p_2 = \beta_2^2$. This implies

$$C_s p_1 + A_s p_2 = \frac{(1+r^2)^2 \beta_2^2}{4mR^2 r^2}$$

and therefore

$$\tilde{h} = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{\beta_2^2}{r^2}\right) + V(r)$$

that corresponds to an integrable system with one degree of freedom.

3. The case $\beta_1 > 0, \beta_2 = 0$ is completely similar to the preceding one with $\widetilde{M}_{ess,\beta} = T^* I$ and

$$\tilde{h} = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{\beta_1^2}{r^2}\right) + V(r) \ .$$

4. In the case $\beta_1 = \beta_2 = 0$ one has $\mathcal{O}_\beta = \mathcal{O}'_\beta = \widetilde{\mathcal{O}_\beta} = \mathrm{pt}$, $\widetilde{M}_{ess,\beta} = T^* I$ and the reduced Hamiltonian function is defined by (7.37).
5. Consider the last case $\beta_1 = \beta_2 > 0$ and show that particles motion occurs now along a two-dimensional sphere $\mathbf{S}^2 \subset \mathbf{S}^3$ that reduces this case to the case $n = 2$, considered above. Indeed, subtracting equations (7.39) from each other one obtains on $\mathcal{O}'_\beta$:

$$\psi_{14}\psi_{23} = \psi_{24}\psi_{13} \ .$$

Recall that initially particles are on the geodesic $x_3 = x_4 = 0$ of the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$ and their initial "collective" momenta (i.e., all momenta except the momentum p_r) are identified with elements of the subspace $\widetilde{\mathfrak{p}} \subset \mathfrak{so}^*(4)$, spanned by Ψ^{ij}, $1 \leqslant i \leqslant 2, 2 \leqslant j \leqslant 4, i < j$. Using the rotation about this geodesic (by the action of the subgroup K_0) one can suppose that $\psi_{14}|_{t=0} = 0$, which corresponds either to $\psi_{24}|_{t=0} = 0$ or to $\psi_{13}|_{t=0} = 0$.
In the latter case $\psi_{14}|_{t=0} = \psi_{13}|_{t=0} = 0$ along the whole K_0-trajectory and one can suppose again $\psi_{24}|_{t=0} = 0$.
In the case $\psi_{14}|_{t=0} = \psi_{24}|_{t=0} = 0$ particles velocities at $t = 0$ are tangent to the 2-sphere $x_1^2+x_2^2+x_3^2 = R^2$, $x_4 = 0$, which implies that both particles will remain on this 2-sphere for all $t > 0$.

This completes the classification of all cases for the Hamiltonian reduction of the two-body problem on the spheres $\mathbf{S}^n$.

7.5.2 Hamiltonian Reduction of the Two-Body Problem on Spaces $\mathbf{H}^2$ and $\mathbf{H}^3$

Again, the classical two-body problem on the hyperbolic spaces $\mathbf{H}^n(\mathbb{R})$ reaches its full generality at $n = 3$.

First consider the case $n = 2$. According to expansion (7.1) the phase space is now

$$T^* I \times T^* \left(\mathbf{H}^2(\mathbb{R})_S\right) ,$$

where $\mathbf{H}^2(\mathbb{R})_S$ is the unit sphere bundle over the hyperbolic plane $\mathbf{H}^2(\mathbb{R})$. Here the group K_0 is trivial, the bundle $\mathbf{H}^2(\mathbb{R})_S$ is diffeomorphic to the group

$O_0(1,2)$ and the space $T^*\left(\mathbf{H}^2(\mathbb{R})_S\right)$ is reduced to an $\mathrm{Ad}^*_{O_0(1,2)}$-orbit (see Remark 4.6).

The $\mathrm{Ad}^*_{O_0(1,2)}$-action conserves the expression $\bar{p}_0^2+\bar{p}_1^2-\bar{p}_2^2$, where $\bar{p}_0, \bar{p}_1, \bar{p}_2$ are linear coordinates on $\mathfrak{so}^*(1,2)$ (see Sect. 7.2.4). A one-sheet hyperboloid, defined by the equation $\bar{p}_0^2+\bar{p}_1^2-\bar{p}_2^2 = c > 0$, represents the one type of $\mathrm{Ad}^*_{O_0(1,2)}$-orbits, diffeomorphic to the manifold $\mathbf{S}^1 \times \mathbb{R}$. Another type is represented by the sheet of a two-sheet hyperboloid, defined by the equation $\bar{p}_0^2+\bar{p}_1^2-\bar{p}_2^2 = c < 0$. Orbits of this type are diffeomorphic to the manifold $\mathbb{R}^2$. The next type consists of two orbits: $\bar{p}_0^2+\bar{p}_1^2 = \bar{p}_2^2$, $\bar{p}_2 > 0$ and $\bar{p}_0^2+\bar{p}_1^2 = \bar{p}_2^2$, $\bar{p}_2 < 0$, diffeomorphic to the manifold $\mathbb{R}^2 \setminus \mathrm{pt}$. The last type consists of the most degenerated orbit: $\bar{p}_0 = 0$, $\bar{p}_1 = 0$, $\bar{p}_2 = 0$.

The reduced Hamiltonian function for $\alpha = m_2/(m_1+m_2)$ is

$$\begin{aligned}\tilde{h} = {} & \frac{(1-r^2)^2}{8mR^2} p_r^2 \\ & + \frac{\bar{p}_0^2}{2(m_1+m_2)R^2} + \frac{1}{2}\left(C_h(r)\bar{p}_1^2 + A_h(r)\bar{p}_2^2 + 2B_h(r)\bar{p}_1\bar{p}_2\right) + V(r)\,.\end{aligned}$$

A suitable pair of functions from the triple $\bar{p}_0, \bar{p}_1, \bar{p}_3$ is local coordinates on a two-dimensional $\mathrm{Ad}^*_{O_0(1,2)}$-orbit and their Poisson brackets are defined from (7.23).

Similar to the spherical case, a two-dimensional invariant submanifold of the reduced space defined by equations $p_1 \equiv p_2 \equiv 0$ corresponds to the bodies motion along a common geodesic. Also, since for $m_1 = m_2 = 2m$, $\alpha = 1/2$ one has $B_h \equiv 0$ there are two other invariant submanifolds in this case. The first one corresponds to a pure transvection and is defined by equations $p_0 \equiv p_2 \equiv 0$ and the second one, defined by $p_0 \equiv p_1 \equiv 0$, corresponds to a pure rotation of the system, both generally with nonconstant velocity.

For the most degenerated orbit one has an integrable reduced Hamiltonian system with one degree of freedom. Now $\widetilde{M}_{ess,0} = T^*I$ and

$$\tilde{h} = \frac{(1-r^2)^2}{8mR^2} p_r^2 + V(r)\,. \tag{7.45}$$

Consider the two-body problem on the space $\mathbf{H}^3(\mathbb{R})$. The corresponding phase space is

$$T^*I \times T^*\left(\mathbf{H}^3(\mathbb{R})_S\right) \tag{7.46}$$

and also it holds $G = O_0(1,3)$, $K_0 = \mathrm{SO}(2)$. Choose the base $\overline{\Psi}_{ij}$, $i \leqslant i < j \leqslant 4$, of the Lie algebra $\mathfrak{so}(1,3)$ in the form (cf. (1.15))

$$\overline{\Psi}_{ij} := \Psi_{ij} = \frac{1}{2}\left(E_{ij} - E_{ji}\right),\ 2 \leqslant i < j \leqslant 4;\ \overline{\Psi}_i := \frac{1}{2}\left(E_{i1} + E_{1i}\right),\ 2 \leqslant i \leqslant 4\,.$$

The commutator relations for this base are

$$\begin{aligned}[\overline{\Psi}_{kj}, \overline{\Psi}_{ml}] &= \frac{1}{2}\left(\delta_{jm}\,\overline{\Psi}_{kl} - \delta_{km}\,\overline{\Psi}_{jl} + \delta_{kl}\,\overline{\Psi}_{jm} - \delta_{jl}\,\overline{\Psi}_{km}\right),\ 2 \leqslant k, j, m, l \leqslant 4, \\ [\overline{\Psi}_{ij}, \overline{\Psi}_k] &= \frac{1}{2}\left(\delta_{jk}\,\overline{\Psi}_i - \delta_{ik}\,\overline{\Psi}_j\right),\ 2 \leqslant i < j \leqslant 4,\ 2 \leqslant k \leqslant 4,\end{aligned} \tag{7.47}$$

$$[\overline{\Psi}_i, \overline{\Psi}_k] = \frac{1}{2}\,\overline{\Psi}_{ik},\; 2 \leqslant i,k \leqslant 4,\; i \neq k\;.$$

Here we suppose as before that $\overline{\Psi}_{kj} = -\,\overline{\Psi}_{jk}$, $2 \leqslant k,j \leqslant 4$.

In notations of Sect. 1.2 elements $\overline{\Psi}_{ij}$, $2 \leqslant i < j \leqslant 4$ span the Lie subalgebra $\mathfrak{k} \cong \mathfrak{so}(3)$ of the algebra $\mathfrak{so}(1,3)$ and elements $\overline{\Psi}_i$, $2 \leqslant i \leqslant 4$ span the subspace $\mathfrak{p}$. The transition from the algebra $\mathfrak{so}(4)$ to the algebra $\mathfrak{so}(1,3)$ is generated by the substitution

$$\Psi_{ij} \to \overline{\Psi}_{ij},\; 2 \leqslant i < j \leqslant 4,\; \Psi_{1i} \to \mathbf{i}\,\overline{\Psi}_i,\; 2 \leqslant i \leqslant 4\;. \tag{7.48}$$

Now the subalgebra $\mathfrak{k}_0$ from expansion (1.2) is generated by $\overline{\Psi}_{34}$, the subalgebra $\mathfrak{a}$ is generated by $\overline{\Psi}_2$, the subspace $\mathfrak{p}_{2\lambda}$ is spanned by elements $\overline{\Psi}_3, \overline{\Psi}_4$ and the subspace $\mathfrak{k}_{2\lambda}$ is spanned by elements $\overline{\Psi}_{23}, \overline{\Psi}_{24}$, cf. (3.31).

Let $\overline{\psi}_{ij}$, $2 \leqslant i < j \leqslant 4$, $\overline{\psi}_i$, $2 \leqslant i \leqslant 4$ be coordinates in $\mathfrak{so}^*(1,3)$, corresponding to the base $\overline{\Psi}^{ij}$, $2 \leqslant i < j \leqslant 4$, $\overline{\Psi}^i$, $2 \leqslant i \leqslant 4$, dual to the base $\overline{\Psi}_{ij}$, $2 \leqslant i < j \leqslant 4$, $\overline{\Psi}_i$, $2 \leqslant i \leqslant 4$. Using substitution (7.48) one obtains from (7.40) the following $\mathrm{Ad}^*_{\mathrm{O}_0(1,3)}$-invariants:

$$\begin{aligned} \overline{\mathrm{J}}_1 &= \overline{\psi}_{23}^2 + \overline{\psi}_{24}^2 + \overline{\psi}_{34}^2 - \overline{\psi}_2^2 - \overline{\psi}_3^2 - \overline{\psi}_4^2,\\ \overline{\mathrm{J}}_2 &= \overline{\psi}_{23}\,\overline{\psi}_4 - \overline{\psi}_3\,\overline{\psi}_{24} + \overline{\psi}_2\,\overline{\psi}_{34}\;. \end{aligned} \tag{7.49}$$

It can be easily verified by direct calculations that the rank of $\mathrm{Ad}^*_{\mathrm{O}_0(1,3)}$-action equals 4 at all points of $\mathfrak{so}^*(1,3)$ except 0. Therefore, any connected component of a level set in $\mathfrak{so}^*(1,3)$, defined by $\overline{\mathrm{J}}_1 = \beta_1$, $\overline{\mathrm{J}}_2 = \beta_2$, is a 4-dimensional $\mathrm{Ad}^*_{\mathrm{O}_0(1,3)}$-orbit for $\beta_1, \beta_2 \in \mathbb{R}, \beta_1^2 + \beta_2^2 \neq 0$. The same is also valid for the level set $\overline{\mathrm{J}}_1 = \overline{\mathrm{J}}_2 = 0$, after excluding the point $0 \in \mathfrak{so}^*(1,3)$.

The subspace $\operatorname{ann}\mathfrak{k}_0$ is defined by the condition $\overline{\psi}_{34} = 0$. Similarly to the spherical case one gets

$$\begin{aligned} &\bar{p}_0 = -2\,\overline{\psi}_2,\; \bar{p}_1 = 4(\overline{\psi}_3^2 + \overline{\psi}_4^2),\; \bar{p}_2 = 4(\overline{\psi}_{23}^2 + \overline{\psi}_{24}^2),\\ &\bar{p}_3 = -4\left(\overline{\psi}_3\,\overline{\psi}_{23} + \overline{\psi}_4\,\overline{\psi}_{24}\right),\; \bar{p}_\Box = 4\left(\overline{\psi}_3\,\overline{\psi}_{24} - \overline{\psi}_4\,\overline{\psi}_{23}\right) \end{aligned}$$

on $\operatorname{ann}\mathfrak{k}_0$ and therefore

$$\bar{p}_0^2 + \bar{p}_1 - \bar{p}_2 = -4\,\overline{\mathrm{J}}_1,\; \bar{p}_\Box = -4\,\overline{\mathrm{J}}_2\;. \tag{7.50}$$

Let $\mathcal{O}_\beta$ be some $\mathrm{Ad}^*_{\mathrm{O}_0(1,3)}$-orbit, for which $\overline{\mathrm{J}}_1 = \beta_1$, $\overline{\mathrm{J}}_2 = \beta_2$. Consider a point $z \in \mathcal{O}'_\beta := \mathcal{O}_\beta \cap \operatorname{ann}\mathfrak{k}_0$ different from the origin with coordinates $\left(\overline{\psi}_2, \overline{\psi}_3, \overline{\psi}_4, \overline{\psi}_{23}, \overline{\psi}_{24}, \overline{\psi}_{34} = 0\right)$. This point is regular for the map $\mathfrak{so}^*(1,3) \to \mathbb{R}^2$ defined by $(\overline{\mathrm{J}}_1, \overline{\mathrm{J}}_2)$ and a vector

$$Y = y_2\,\overline{\Psi}^2 + y_3\,\overline{\Psi}^3 + y_4\,\overline{\Psi}^4 + y_{23}\,\overline{\Psi}^{23} + y_{24}\,\overline{\Psi}^{24} + y_{34}\,\overline{\Psi}^{34} \in T_z\mathfrak{so}^*(1,3)$$

belongs to $T_z\mathcal{O}_\beta$ iff

$$\begin{aligned} &\overline{\psi}_{23}\,y_{23} + \overline{\psi}_{24}\,y_{24} - \overline{\psi}_3\,y_3 - \overline{\psi}_4\,y_4 = \overline{\psi}_2\,y_2,\\ &\overline{\psi}_4\,y_{23} - \overline{\psi}_3\,y_{24} - \overline{\psi}_{24}\,y_3 + \overline{\psi}_{23}\,y_4 = -\,\overline{\psi}_2\,y_{34}\;. \end{aligned} \tag{7.51}$$

Since $\dim \operatorname{ann} \mathfrak{k}_0 = 5$, the orbit $\mathcal{O}_\beta$ is not transversal to $\operatorname{ann} \mathfrak{k}_0$ at the point z iff $T_z\mathcal{O}_\beta \subset \operatorname{ann} \mathfrak{k}_0$. Via coordinates the last inclusion means that the system (7.51) implies $y_{34} = 0$. For $\overline{\psi}_2 = 0$ the variable y_{34} has an arbitrary value and $\mathcal{O}_\beta$ is transversal to $\operatorname{ann} \mathfrak{k}_0$ at the point z. For $\overline{\psi}_2 \neq 0$ the independent variables y_{23}, y_{24}, y_3, y_4 uniquely define values of y_2 and y_{34}. Therefore, the system (7.51) implies $y_{34} = 0$ only if $\overline{\psi}_4 = \overline{\psi}_3 = \overline{\psi}_{24} = \overline{\psi}_{23} = 0$. This means that $\beta_1 = -\overline{\psi}_2^2 < 0$, $\beta_2 = 0$. Thus, for $\beta_1 \geqslant 0$ or $\beta_2 \neq 0$ the orbit $\mathcal{O}_\beta$, different from the origin, satisfies Assumptions 4.2 and 4.3.

Since

$$[\overline{\Psi}_{34}, \overline{\Psi}_{23}] = -\frac{1}{2}\,\overline{\Psi}_{24},\ [\overline{\Psi}_{34}, \overline{\Psi}_{24}] = \frac{1}{2}\,\overline{\Psi}_{23},\ [\overline{\Psi}_{34}, \overline{\Psi}_{4}] = \frac{1}{2}\,\overline{\Psi}_{3},\ [\overline{\Psi}_{34}, \overline{\Psi}_{3}] = -\frac{1}{2}\,\overline{\Psi}_{4}\ ,$$

the $\operatorname{Ad}^*_{K_0}$-action is the synchronous orthogonal rotation in two 2-planes in the space $\mathfrak{so}^*(1,3)$, generated by pairs $\overline{\Psi}_{23}, \overline{\Psi}_{24}$ and $\overline{\Psi}_3, \overline{\Psi}_4$ respectively.

1. Let us show that in the case $\beta_2 = 0$ a particles motion occurs along a two-dimensional hyperbolic subspace $\mathbf{H}^2(\mathbb{R}) \subset \mathbf{H}^3(\mathbb{R})$ that reduces this case to the case $n = 2$, considered above.
   Indeed, now it holds $\overline{\psi}_{23}\,\overline{\psi}_4 = \overline{\psi}_{24}\,\overline{\psi}_3$. Consider the space $\mathbf{H}^3(\mathbb{R})$ as the sheet with $x_1 \geqslant 1$ of the two-sheet hyperboloid
   $$x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2$$
   in the Minkowski space, see Sect. 1.3.3. Suppose that initially particles are on the geodesic $x_3 = x_4 = 0$. Their initial "collective" momenta are identified with elements of the subspace $\widetilde{\mathfrak{p}} \subset \mathfrak{so}^*(1,3)$, generated by $\overline{\Psi}^{23}, \overline{\Psi}^{24}, \overline{\Psi}^{i}$, $2 \leqslant i \leqslant 4$. Using the rotation about this geodesic (the action of the subgroup K_0) one can suppose $\overline{\psi}_4\big|_{t=0} = 0$ that corresponds either to $\overline{\psi}_{24}\big|_{t=0} = 0$ or to $\overline{\psi}_3\big|_{t=0} = 0$.
   In the latter case $\psi_4|_{t=0} = \psi_3|_{t=0} = 0$ along the whole K_0-trajectory and one can suppose again $\psi_{24}|_{t=0} = 0$.
   In the case $\psi_4|_{t=0} = \psi_{24}|_{t=0} = 0$ particles velocities at $t = 0$ are tangent to the hyperbolic subspace $\mathbf{H}^2(\mathbb{R})$, defined by the equation $x_4 = 0$, and both particles will remain on it for all $t > 0$.
2. Let now $\mathcal{O}_\beta$ be an $\operatorname{Ad}^*_{K_0}$-orbit for $\beta_2 \neq 0$. Then $\mathcal{O}'_\beta = \mathcal{O}_\beta \cap \operatorname{ann} \mathfrak{k}_0$ is a smooth submanifold in $\mathfrak{so}^*(1,3)$. Due to $\beta_2 \neq 0$ the submanifold $\mathcal{O}'_\beta$ does not contain points with coordinates $\overline{\psi}_3 = \overline{\psi}_4 = \overline{\psi}_{23} = \overline{\psi}_{24} = 0$ and the $\operatorname{Ad}^*_{K_0}$-action on $\mathcal{O}'_\beta$ is free that implies the Assumption 4.4. Thus, the factor space $\widetilde{\mathcal{O}}_\beta := \mathcal{O}'_\beta / \operatorname{Ad}^*_{K_0}$ is a smooth manifold.
   An every $\operatorname{Ad}^*_{K_0}$-orbit in $\mathcal{O}'_\beta$ contains a unique point with coordinates $\overline{\psi}_{24} = 0, \overline{\psi}_{23} > 0$. For this point it holds $\overline{\psi}_4 = \beta_2 / \overline{\psi}_{23}$ and
   $$\overline{\psi}_2^2 + \overline{\psi}_3^2 = \overline{\psi}_{23}^2 - \frac{\beta_2^2}{\overline{\psi}_{23}^2} - \beta_1\ . \tag{7.52}$$
   The right hand side in (7.52) is nonnegative for

$$\overline{\psi}_{23} \geqslant \frac{1}{\sqrt{2}}\sqrt{\beta_1 + \sqrt{\beta_1^2 + 4\beta_2^2}} > 0.$$

Also, for these values of $\overline{\psi}_{23}$ it infinitely increases in a monotone way. Therefore, (7.52) and the inequality $\overline{\psi}_{23} > 0$ define the surface in the subspace

$$\operatorname{span}(\overline{\Psi}^2, \overline{\Psi}^3, \overline{\Psi}^{23}) \subset \mathfrak{so}^*(1,3),$$

diffeomorphic to $\mathbb{R}^2$, with global coordinates $\overline{\psi}_2, \overline{\psi}_3$.
Thus, $\mathrm{Ad}^*_{K_0}$-orbits in $\mathcal{O}'_\beta$ for $\beta_2 \neq 0$ are in one-to-one correspondence with points of this surface and due to Theorem 4.6 the reduced phase space for (7.46) is diffeomorphic to $T^*I \times \mathbb{R}^2$. The reduced Hamiltonian function is given by (7.22), where due to (7.50) it holds

$$\bar{p}_0^2 + \bar{p}_1 - \bar{p}_2 = -4\beta_1,\ \bar{p}_1\bar{p}_2 - \bar{p}_3^2 = \bar{p}_\square^2 = 16\beta_2^2.$$

This completes the classification of all cases for the Hamiltonian reduction of the two-body problem on the hyperbolic space $\mathbf{H}^n(\mathbb{R})$.

Any two functions from the quadruple p_0, p_1, p_2, p_3 are independent on a reduced phase space. The commutator relations for these functions are in (7.23). Again the reduced Hamiltonian system has two degrees of freedom and the problem of its integrability for nontrivial potentials is open.

8

Quasi-Exactly Solvability of the Quantum Mechanical Two-Body Problem on Spheres

It is assumed that a quantum mechanical system is exactly solvable if its energy levels and corresponding eigenstates are known in explicit form. Usually, this solvability is a consequence of existing of many symmetries for this system.

Sometimes only a part of the spectrum and corresponding eigenvectors are known in explicit form. Such systems are called *quasi-exactly solvable* [195, 196, 197]. In this section we will show that the two-body quantum mechanical problem on spheres with Coulomb and oscillator potentials is quasi-exactly solvable. For other compact two-point homogeneous spaces corresponding problems lead to the Heun equation.

First we shall describe the common scheme for finding an infinite sequence of eigenvalues for two-body Hamiltonians on a compact two-point homogeneous space Q without detailed description of the self-adjoint extension of formal Hamiltonians, which will be done later.

In notations of Chap. 5 we will suppose below that $m_1\alpha = m_2\beta$. Operators $D_0^2, D_1, D_2, \ldots$ from the expression for the two-body Hamiltonian H (one of (5.22)–(5.25)) act on the space $Q_S \cong G/K_0$. Let $\psi_D \in \mathcal{L}^2(G/K_0)$ be their common eigenfunction. Then one can try to find an eigenfunction of the Hamiltonian H in the form $f(r)\psi_D$, where r is a radial coordinate on the corresponding interval I from expansion (5.2). For example one needs a common eigenvector of operators D_0^2, D_1, D_2, D_3 for the Hamiltonian (5.24) and $D_0^2, D_1^2, D_2^2, \{D_1, D_2\}$ for the Hamiltonian (5.25). Note that for particles with equal masses these conditions could be made less restrictive: ψ_D should not be an eigenvector of operators D_3, D_6 for the Hamiltonians (5.22), (5.24), or $D_3, \{D_4, D_5\}$ for (5.23), or $\{D_1, D_2\}$ for (5.25), since in this case $B_s \equiv F_s \equiv 0$.

Evidently, for such choice of ψ_D the following stationary Schrödinger equation

$$H\left(f(r)\psi_D\right) = Ef(r)\psi_D \tag{8.1}$$

is equivalent to the spectral problem for an ordinary differential equation for a function $f(r)$ and an energy level E (in other words to a one-dimensional Shrödinger equation).

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The search for all common eigenfunctions of operators $D_0^2, D_1, D_2, \ldots$ is not an easy problem. Note, however, that due to the compactness of Q_S there is a trivial common eigenfunction of $D_0^2, D_1, D_2, \ldots$, universal for all Q. It is a function $\psi_D = \text{const} \neq 0$ with the common eigenvalue equals 0.

In following sections we shall find all common eigenfunctions of operators $D_0^2, D_1, D_2, \ldots$ for the spheres $\mathbf{S}^n$ and projective spaces $\mathbf{P}^n(\mathbb{R})$. Then we shall return to corresponding one-dimensional Shrödinger equations. This chapter is based upon [170].

Recall first of all basic facts concerning linear representations of compact Lie groups.

8.1 Regular Representations of Compact Lie Groups

Let G be a compact connected Lie group and μ be a biinvariant positive measure on G, unique up to an arbitrary multiplicative constant [88]. Let $\mathcal{L}^2(G, d\mu)$ be a Hilbert space of measurable complex-valued functions on G, square integrable w.r.t. the measure μ. Define two unitary left representations of G in the space $\mathcal{L}^2(G, d\mu)$. The *left regular representation* T^l acts by the left shifts

$$\left(T_q^l f\right)(g) = f(q^{-1}g),\ q, g \in G, f \in \mathcal{L}^2(G, d\mu)$$

and the *right regular representation* T^r acts by the right shifts

$$\left(T_q^r f\right)(g) = f(gq),\ q, g \in G, f \in \mathcal{L}^2(G, d\mu)\ .$$

Evidently, these representations are equivalent with the intertwining operator $f(g) \to f(g^{-1})$. It is well known that these representations can be decomposed into direct sums of finite-dimensional unitary irreducible representations (irreps). Each of these irreps is contained in T^l or T^r with a multiplicity equal to its dimension and any linear irreducible representation of G is equivalent to an irreps from this sum [13, 199].

Let T_ℓ be a full system of unitary irreps for G in spaces U_ℓ, $\ell = 1, 2, \ldots$. Choose in every U_ℓ an orthonormal base $(e_{\ell,k})_{k=1}^{d_\ell}$, $d_\ell := \dim_{\mathbb{C}} U_\ell$. Define matrix elements $t_{\ell,k}^i$ of operators T_q^r by the equation $T_q^r e_{\ell,k} =: t_{\ell,k}^i(q) e_{\ell,i}$ or equivalently by $t_{\ell,k}^i(q) := \langle e_{\ell,i}, T_q^r e_{\ell,k}\rangle_{U_\ell}$, $q \in G$. Since

$$t_{\ell,k}^i(gq)e_{\ell,i} = T_g^r T_q^r e_{\ell,k} = t_{\ell,i}^j(g) t_{\ell,k}^i(q) e_{\ell,j},\ g, q \in G\ ,$$

one has

$$t_{\ell,k}^i(gq) = t_{\ell,j}^i(g) t_{\ell,k}^j(q)\ . \tag{8.2}$$

Therefore, the subspace $\mathcal{R}_{\ell,i} \subset \mathcal{L}^2(G, d\mu)$, spanned by functions $(t_{\ell,j}^i(g))_{j=1}^{d_\ell}$, is invariant under operators T_q^r and the representation $T^r|_{\mathcal{R}_{\ell,i}}$ is equivalent to T_ℓ. On the other hand, formula (8.2) implies that the subspace $\mathcal{L}_{\ell,j} \subset \mathcal{L}^2(G, d\mu)$, spanned by functions $(t_{\ell,j}^i(g))_{i=1}^{d_\ell}$, is invariant under operators T_q^l and the representation $T^l\big|_{\mathcal{L}_{\ell,j}}$ is again equivalent to T_ℓ. The functions $(t_{\ell,j}^i(g))_{i,j=1}^{d_\ell}$, $\ell = 1, 2, \ldots$ form an orthogonal base in the space $\mathcal{L}^2(G, d\mu)$ [13, 88, 199] and

$$\|t^i_{\ell,j}\|^2_{\mathcal{L}^2(G,d\mu)} = \frac{\mu(G)}{d_\ell} .$$

Thus, the space

$$\mathcal{T}_\ell := \bigoplus_{i=1}^{d_\ell} \mathcal{R}_{\ell,i} = \bigoplus_{j=1}^{d_\ell} \mathcal{L}_{\ell,j}$$

is invariant under representations T^r and T^l. The representation T^r intermixes spaces $\mathcal{L}_{\ell,j}$ of representations T^l and vise versa the representation T^l intermixes spaces $\mathcal{R}_{\ell,i}$ of representations T^r. The space $\mathcal{L}^2(G, d\mu)$ of representations T^r and T^l is expanded into irreps as follows

$$\mathcal{L}^2(G, d\mu) = \bigoplus_{\ell} \mathcal{T}_\ell = \bigoplus_{\ell} \bigoplus_{i=1}^{d_\ell} \mathcal{R}_{\ell,i} = \bigoplus_{\ell} \bigoplus_{j=1}^{d_\ell} \mathcal{L}_{\ell,j} .$$

For a Lie subgroup K_0 of the group G the subspace $\mathcal{L}^2\left(G, K_0, d\mu\right) \subset \mathcal{L}^2(G, d\mu)$, consisting of functions invariant w.r.t. all right K_0-shifts on G, is invariant w.r.t left G-shifts. Therefore, there are only two possibilities:

$$\mathcal{L}_{\ell,j} \cap \mathcal{L}^2\left(G, K_0, d\mu\right) = \mathcal{L}_{\ell,j} \text{ and } \mathcal{L}_{\ell,j} \cap \mathcal{L}^2\left(G, K_0, d\mu\right) = 0 .$$

The consideration above implies the following proposition.

Proposition 8.1. *Let*

$$\widetilde{\mathcal{T}}_\ell := \mathcal{T}_\ell \cap \mathcal{L}^2\left(G, K_0, d\mu\right), \, \tilde{d}_\ell := \dim_{\mathbb{C}} \left(\mathcal{R}_{\ell,i} \cap \mathcal{L}^2\left(G, K_0, d\mu\right)\right) .$$

Evidently, the value $\tilde{d}_\ell$ does not depend on $i = 1, \ldots, d_\ell$. The representation $T^l\big|_{\widetilde{\mathcal{T}}_\ell}$ is expanded into the direct sum of equivalent irreps in spaces $\mathcal{L}^{K_0}_{\ell,k}$, $k = 1, \ldots, \tilde{d}_\ell$, which are among of $\mathcal{L}_{\ell,j}$. On the other hand

$$\widetilde{\mathcal{T}}_\ell = \bigoplus_{i=1}^{d_\ell} \widetilde{\mathcal{R}}_{\ell,i} ,$$

where the spaces $\widetilde{\mathcal{R}}_{\ell,i}$, $i = 1, \ldots, d_\ell$, are isomorphic to each other.

Let now G be the identity component of the isometry group for a two-point homogeneous space Q and as above K_0 be its stationary subgroup corresponding to an element from Q_S. Operators D_i, constructed in Chap. 3, are polynomial w.r.t. infinitesimal generators of right G-shifts. Therefore, they conserve the spaces $\widetilde{\mathcal{T}}_\ell$ and generally intermix its direct summands $\mathcal{L}^{K_0}_{\ell,k}$, $k = 1, \ldots, \tilde{d}_\ell$ with constant ℓ and different k. On the other hand they act in spaces $\widetilde{R}_{\ell,i}$ and their actions are isomorphic to each other for constant ℓ and different $i = 1, \ldots, d_\ell$.

8.2 Common Eigenfunctions of Operators D_i for Spheres $\mathbf{S}^n$ and Projective Spaces $\mathbf{P}^n(\mathbb{R})$

In the present section we follow the consideration from [170] that generalizes results for cases $Q = \mathbf{S}^2$ and $Q = \mathbf{S}^3$ found in [162] and [184] respectively.

First of all one can get some a priori information on eigenvalues of common eigenfunctions for operators D_0^2, D_1, D_2, D_3 in the case $Q = \mathbf{S}^n$, $n \geqslant 3$ or for operators $D_0^2, D_1^2, D_2^2, \{D_1, D_2\}$ in the case $Q = \mathbf{S}^2$.

Proposition 8.2. *Let ψ_D be a common eigenfunction for operators D_0^2, D_1', D_2', D_3' with eigenvalues $\delta_0, \delta_1, \delta_2$ and δ_3 respectively, where $D_i' = D_i$, $i = 1, 2, 3$ for $Q = \mathbf{S}^n$, $n \geqslant 3$ and $D_1' = D_1^2$, $D_2' = D_2^2$, $D_3' = \frac{1}{2}\{D_1, D_2\}$ for $Q = \mathbf{S}^2$. Then*

1. *$\delta_1 = \delta_2$ and $\delta_3 = 0$;*
2. *$D_0\psi_D$ is an eigenfunction for operators D_0^2, D_1', D_2', D_3' with the same eigenvalues $\delta_0, \delta_1, \delta_2$ and δ_3 respectively;*
3. *if $D_0\psi_D \not\sim \psi_D$, then $D_0\psi_D \pm \sqrt{\delta_0}\psi_D$ is an eigenfunction for operators D_0, D_1', D_2', D_3';*
4. *if $D_0\psi_D \sim \psi_D$, then either $D_0\psi_D = 0$ or $\delta_1 = \delta_2 = (n-1)(n-3)/4$.*

Proof. Consider the case $Q = \mathbf{S}^n$, $n \geqslant 3$. Relations $[D_0, D_3] = D_1 - D_2$ and $[D_1, D_2] = -2\{D_0, D_3\}$ (see (3.35)) imply

$$[D_0, D_3]\psi_D = \delta_3 D_0\psi_D - D_3 D_0\psi_D = (D_1 - D_2)\psi_D = (\delta_1 - \delta_2)\psi_D \; ,$$
$$\delta_3 D_0\psi_D + D_3 D_0\psi_D = \{D_0, D_3\}\psi_D = -\frac{1}{2}[D_1, D_2]\psi_D = 0 \; . \tag{8.3}$$

The last two equations lead to

$$2\delta_3 D_0\psi_D = (\delta_1 - \delta_2)\psi_D \; . \tag{8.4}$$

If $\delta_3 \neq 0$, then the last equation implies $D_0\psi_D \sim \psi_D$ and the relation $[D_0, D_1] = -2D_3$ gives $\delta_3\psi_D = D_3\psi_D = -\frac{1}{2}[D_0, D_1]\psi_D = 0$. Thus, $\delta_3 = 0$ and (8.4) implies $\delta_1 = \delta_2$ that proves the first claim of the proposition.

Now from (8.3) one gets $D_3 D_0\psi_D = 0$ and the first two relations (3.35) imply $D_1 D_0\psi_D = D_2 D_0\psi_D = \delta_1 D_0\psi_D$. The relation $D_0^2 D_0\psi_D = \delta_0 D_0\psi_D$ is evident that completes the proof of the second claim.

The relation $D_0^2\psi_D = \delta_0\psi_D$ is equivalent to $\left(D_0 + \sqrt{\delta_0}\,\mathrm{id}\right)\left(D_0 - \sqrt{\delta_0}\,\mathrm{id}\right)\psi_D = 0$. Now if $D_0\psi_D \neq \sqrt{\delta_0}\psi_D$, then $\psi_D^- := \left(D_0 - \sqrt{\delta_0}\,\mathrm{id}\right)\psi_D$ is an eigenfunction for the operator D_0. The function ψ_D^- is also an eigenfunction for operators D_1, D_2, D_3 due to the second claim. The consideration for the function $\psi_D^+ := \left(D_0 + \sqrt{\delta_0}\,\mathrm{id}\right)\psi_D$ is completely similar. Thus, the third claim is proved.

Assume now $D_0\psi_D = \delta_0'\psi_D$. Then the last relation from (3.35) gives

$$2\delta_0'\delta_2\psi_D = \frac{1}{2}(n-1)(n-3)\delta_0'\psi_D \; .$$

This means either $\delta_0' = 0$ or $\delta_1 = \delta_2 = (n-1)(n-3)/4$ that proves the last claim.

The case $Q = \mathbf{S}^2$ is completely similar. □

In this section, we shall use notations of Sect. 8.1 for $G = \mathrm{SO}(n+1)$ and $K_0 = \mathrm{SO}(n-1)$. Below we mean by the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ the following set

$$\mathfrak{so}(n+1,\mathbb{C}) = \left(A \in \mathfrak{gl}(n+1,\mathbb{C})|\ A + A^T = E\right) . \tag{8.5}$$

From now we shall treat complex spaces $R_{\ell,i}$ as simple left modules over $\mathfrak{g}^{\mathbb{C}}$. Their subspaces $\widetilde{R}_{\ell,i}$ consist of elements annulled by the subalgebra $\mathfrak{k}_0^{\mathbb{C}} \cong \mathfrak{so}(n-1,\mathbb{C}) \subset \mathfrak{g}^{\mathbb{C}}$.

The classification of such modules based on the notion of a dominant weight is well known [53, 60] (see also appendix C for a brief description). In order to apply this theory one should use a form of $\mathfrak{so}(n+1,\mathbb{C})$, described in appendix C and different from (8.5). Besides, since $\mathfrak{B}_k := \mathfrak{so}(2k+1,\mathbb{C})$ and $\mathfrak{D}_k := \mathfrak{so}(2k,\mathbb{C})$ are different series of simple complex Lie algebras, we shall consider cases of odd and even n separately.

8.2.1 The Case $n = 2k$

In this section, we shall use notations from Appendix C.1. In particular, by $\mathfrak{B}_k$ we mean the set (C.1). First of all we shall construct the isomorphism $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{B}_k$ in explicit form.

Let

$$J_{2k+1} = \begin{pmatrix} \frac{1}{\sqrt{2}}E_k & 0 & \frac{1}{\sqrt{2}}S_k \\ 0 & 1 & 0 \\ \frac{\mathbf{i}}{\sqrt{2}}S_k & 0 & \frac{-\mathbf{i}}{\sqrt{2}}E_k \end{pmatrix} \in \mathrm{GL}(2k+1,\mathbb{C}) .$$

It is easily verified that

$$J_{2k+1}S_{2k+1}J_{2k+1}^T = E_{2k+1} .$$

Therefore, the equation $A^T S_{2k+1} + S_{2k+1}A = 0$ for $A \in \mathfrak{gl}(2k+1,\mathbb{C})$ is equivalent to the equation $B^T + B = 0$, where $B := \left(J_{2k+1}^T\right)^{-1} A J_{2k+1}^T$. Thus, the map

$$B \to J_{2k+1}^T B \left(J_{2k+1}^T\right)^{-1} \tag{8.6}$$

is the isomorphism between $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{B}_k$.

Let

$$C = \begin{pmatrix} 0 & \alpha & A_- & a & A_+ \\ -\alpha & 0 & B_- & b & B_+ \\ -A_-^T & -B_-^T & & & \\ -a & -b & & C' & \\ -A_+^T & -B_+^T & & & \end{pmatrix} \in \mathfrak{g} ,$$

where

$$A_- = \left(a_{-(k-1)}, \dots, a_{-1}\right), \ A_+ = \left(a_1, \dots, a_{k-1}\right), \ B_- = \left(b_{-(k-1)}, \dots, b_{-1}\right),$$
$$B_+ = \left(b_1, \dots, b_{k-1}\right), \ a_i, b_i, a, b \in \mathbb{R}, \ C' \in \mathfrak{so}(2k-1) .$$

Move the second row and the second column of the matrix C to the last positions. This gives the matrix

$$\widetilde{C} = \begin{pmatrix} 0 & A & a & A_+ & \alpha \\ -A_-^T & & & & -B_-^T \\ -a & & \widetilde{C}' & & -b \\ -A_+^T & & & & -B_+^T \\ -\alpha & B_- & b & B_+ & 0 \end{pmatrix} \in \mathfrak{so}(2k+1),\ \widetilde{C}' \in \mathfrak{so}(2k-1)\ .$$

The transformation (8.6) now gives for $\widehat{C} := J_{2k+1}^T \widetilde{C} \left(J_{2k+1}^T\right)^{-1}$ the expression

$$\widehat{C} = \frac{1}{2}\begin{pmatrix} -2\mathbf{i}\alpha & Z_- - \mathbf{i}Z_+S_{k-1} \ \sqrt{2}z \ Z_-S_{k-1} + \mathbf{i}Z_+ & 0 \\ -\overline{Z}_-^T - \mathbf{i}S_{k-1}\overline{Z}_+^T & & -Z_-^T - \mathbf{i}S_{k-1}Z_+^T \\ -\sqrt{2}\bar{z} & \widehat{C}' & -\sqrt{2}z \\ -S_{k-1}\overline{Z}_-^T + \mathbf{i}\overline{Z}_+^T & & -S_{k-1}Z_-^T + \mathbf{i}Z_+^T \\ 0 & \overline{Z}_- - \mathbf{i}\overline{Z}_+S_{k-1} \ \sqrt{2}\bar{z} \ \overline{Z}_-S_{k-1} + \mathbf{i}\overline{Z}_+ & 2\mathbf{i}\alpha \end{pmatrix},$$

where $Z_- := A_- + \mathbf{i}B_-$, $Z_+ := A_+ + \mathbf{i}B_+$, $z := a + \mathbf{i}b$, $\widehat{C}' \in \mathfrak{B}_{k-1}$. Let us identify Lie algebras $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{B}_k$ through the map $C \to \widehat{C}$. Due to the definition of Ψ_{ij} in Sect. 1.3.3 one gets the following formulas

$$\Psi_{12} = \frac{\mathbf{i}}{2}F_{kk},\ \Psi_{1,k+2} = \frac{1}{2\sqrt{2}}\left(F_{k0} - F_{0k}\right),\ \Psi_{2,k+2} = -\frac{\mathbf{i}}{2\sqrt{2}}\left(F_{k0} + F_{0k}\right)\ ,$$
$$\Psi_{1i} = \frac{1}{4}\left(F_{kj} + F_{k,-j} + F_{-kj} + F_{-k,-j}\right),\ j = i-k-2,\ 3 \leqslant i \leqslant k+1\ ,$$
$$\Psi_{1i} = \frac{\mathbf{i}}{4}\left(F_{kj} - F_{k,-j} + F_{-kj} - F_{-k,-j}\right),\ j = i-k-2,\ k+3 \leqslant i \leqslant 2k+1\ ,$$
$$\Psi_{2i} = \frac{\mathbf{i}}{4}\left(F_{-kj} + F_{-k,-j} - F_{kj} - F_{k,-j}\right),\ j = i-k-2,\ 3 \leqslant i \leqslant k+1\ ,$$
$$\Psi_{2i} = \frac{1}{4}\left(F_{kj} - F_{k,-j} + F_{-k,-j} - F_{-kj}\right),\ j = i-k-2,\ k+3 \leqslant i \leqslant 2k+1\ ,$$

which imply

$$\begin{aligned} D_1 &= \frac{1}{2}\left(F_{k0} - F_{0k}\right)^2 + \frac{1}{2}\sum_{j=1}^{k-1}\left\{F_{-kj} + F_{kj}, F_{k,-j} + F_{-k,-j}\right\}, \\ D_2 &= -\frac{1}{2}\left(F_{k0} + F_{0k}\right)^2 + \frac{1}{2}\sum_{j=1}^{k-1}\left\{F_{-kj} - F_{kj}, F_{k,-j} - F_{-k,-j}\right\}, \\ D_3 &= \frac{\mathbf{i}}{2}\left(F_{k0}^2 - F_{0k}^2\right) + \mathbf{i}\sum_{j=1}^{k-1}\left(F_{kj}F_{k,-j} - F_{-kj}F_{-k,-j}\right),\ D_0 = -\mathbf{i}F_{kk}\ , \end{aligned} \tag{8.7}$$

for $k \geqslant 2$ and

$$D_0 = -\mathbf{i}F_{11},\ D_1 = \frac{1}{\sqrt{2}}\left(F_{10} - F_{01}\right),\ D_2 = \frac{\mathbf{i}}{\sqrt{2}}\left(F_{10} + F_{01}\right)$$

for $k = 1$.

Since the case $k = 1$ does not fit the general scheme due to the triviality of the group K_0, we assume from now $k \geqslant 2$. The case $k = 1$ will be considered below.

Let $\mathcal{R}_{\ell,i}$ be the space $V_{\mathfrak{B}_k}(\lambda)$ for a highest weight (C.4), where $m_i \in \mathbb{Z}_+$ and $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ be a subspace of $V_{\mathfrak{B}_k}(\lambda)$ annulled by the subalgebra $\mathfrak{k}_0^{\mathbb{C}} \cong \mathfrak{B}_{k-1}$. An element $v \in \widetilde{V}_{\mathfrak{B}_k}(\lambda)$, $v \neq 0$ is a highest vector of the trivial one-dimensional $\mathfrak{B}_{k-1}$-module. Then Propositions C.1 and C.2 imply the existence of such numbers $m'_j \in \mathbb{Z}_+$, $j = 1, \ldots, k$, that

$$\begin{gathered} m_k \geqslant m'_k \geqslant m_{k-1} \geqslant \ldots \geqslant m'_2 \geqslant m_1 \geqslant m'_1 \geqslant -m_1 \, , \\ m'_k \geqslant 0 \geqslant m'_{k-1} \geqslant 0 \geqslant \ldots \geqslant m'_2 \geqslant 0 \geqslant |m'_1| \, . \end{gathered}$$

Thus, $m'_j = 0$, $j = 1, \ldots, k-1$ and therefore $m_j = 0$, $j = 1, \ldots, k-2$.

From now till the end of the present subsection suppose

$$\lambda = m_{k-1}\varepsilon_{k-1} + m_k\varepsilon_k, \; m_k \geqslant m_{k-1} \geqslant 0, \; m_k, m_{k-1} \in \mathbb{Z}_+ \, .$$

In this case, Proposition C.1 implies that any module $V_{\mathfrak{D}_k}(m'_k\varepsilon_k) \subset V_{\mathfrak{B}_k}(\lambda)$ contains the unique one-dimensional module $V_{\mathfrak{B}_{k-1}}(0)$. This fact leads to

$$\dim \widetilde{V}_{\mathfrak{B}_k}(\lambda) = m_k - m_{k-1} + 1 \, . \tag{8.8}$$

Thus, from Proposition 8.1 one gets the following expansion [102]:

$$\begin{aligned} &\mathcal{L}^2\left(\mathrm{SO}(2k+1), \mathrm{SO}(2k-1), d\mu\right) \\ &= \bigoplus_{\substack{m_k \geqslant m_{k-1} \\ m_k, m_{k-1} \in \mathbb{Z}_+}} (m_k - m_{k-1} + 1) V_{\mathfrak{B}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right) \, , \end{aligned}$$

where the left hand side is considered as a restriction of the left regular representation for the group $\mathrm{SO}(2k+1)$.

On the other hand, the space

$$\mathcal{L}^2\left(\mathrm{SO}(2k+1), \mathrm{SO}(2k-1), d\mu\right)$$

as a $\mathrm{Diff}_{\mathrm{SO}(2k+1)}(\mathrm{SO}(2k+1)/\,\mathrm{SO}(2k-1))$-module is expanded as

$$\begin{aligned} &\mathcal{L}^2\left(\mathrm{SO}(2k+1), \mathrm{SO}(2k-1), d\mu\right) \qquad (8.9) \\ &= \bigoplus_{\substack{m_k \geqslant m_{k-1} \\ m_k, m_{k-1} \in \mathbb{Z}_+}} \left(\dim V_{\mathfrak{B}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right)\right) \widetilde{V}_{\mathfrak{B}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right), \end{aligned}$$

where $\dim V_{\mathfrak{B}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right)$ is given by (C.8). Let

$$\begin{aligned} D^+ &:= \sum_{j=1}^{k-1} F_{kj}F_{k,-j} + \frac{1}{2}F_{k0}^2, \; D^- := \sum_{j=1}^{k-1} F_{-kj}F_{-k,-j} + \frac{1}{2}F_{0k}^2 \, , \\ \widetilde{C} &:= C|_{\mathcal{L}^2(\mathrm{SO}(2k+1),\mathrm{SO}(2k-1),d\mu)} = F_{kk}^2 + \{F_{k0}, F_{0k}\} \\ &\quad + \sum_{j=1}^{k-1} \left(\{F_{kj}, F_{jk}\} + \{F_{k,-j}, F_{-jk}\}\right) \end{aligned}$$

be operators from $\mathcal{L}^2\left(\mathrm{SO}(2k+1), \mathrm{SO}(2k-1), d\mu\right)$, where C is the universal Casimir operator (C.6). According to the general agreement made in Chap. 3 we consider elements F_{kk}, D^+, D^- from the complexification of the algebra $U\left(\mathfrak{so}(2k+1, \mathbb{C})\right)^{\mathrm{SO}(2k-1)}$ modulo the ideal

$$\left(U\left(\mathfrak{so}(2k+1, \mathbb{C})\right)\mathfrak{so}(2k-1, \mathbb{C})\right)^{\mathrm{SO}(2k-1)} \subset U\left(\mathfrak{so}(2k+1, \mathbb{C})\right)^{\mathrm{SO}(2k-1)},$$

i.e., as elements of the algebra $\mathrm{Diff}_{\mathrm{SO}(2k+1)}\left(\mathrm{SO}(2k+1)/\,\mathrm{SO}(2k-1)\right)^{\mathbb{C}}$.

Due to (C.3) and (C.9), the operator D^+ "raises" weight subspaces of $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ and the operator D^- "lowers" them.

Since $[F_{kj}, F_{k,-j}] = [F_{-kj}, F_{-k,-j}] = 0$, one gets the following relations

$$\begin{aligned} D_1 &= D^+ + D^- + \frac{1}{2}\left(F_{kk}^2 - \widetilde{C}\right),\ D_2 = -D^+ - D^- + \frac{1}{2}\left(F_{kk}^2 - \widetilde{C}\right), \\ D_3 &= \mathbf{i}\left(D^+ - D^-\right), D^+ = \frac{1}{4}\left(D_1 - D_2\right) - \frac{\mathbf{i}}{2}D_3\,, \\ D^- &= \frac{1}{4}\left(D_1 - D_2\right) + \frac{\mathbf{i}}{2}D_3,\ \widetilde{C} = -D_0^2 - D_1 - D_2\,. \end{aligned} \tag{8.10}$$

Commutator relations (3.35) now give

$$[F_{kk}, D^+] = 2D^+,\ [F_{kk}, D^-] = -2D^-, \tag{8.11}$$

$$[D^+, D^-] = -\frac{1}{2}F_{kk}^3 + \frac{1}{2}\widetilde{C}F_{kk} + \frac{1}{4}(2k-1)(2k-3)F_{kk}\,. \tag{8.12}$$

Formulas (C.5) and (C.7) imply

$$\left.\widetilde{C}\right|_{\widetilde{V}_{\mathfrak{B}_k}(\lambda)} = \left(\left(k + m_k - \frac{1}{2}\right)^2 + \left(k + m_{k-1} - \frac{3}{2}\right)^2 - \left(k - \frac{1}{2}\right)^2 - \left(k - \frac{3}{2}\right)^2\right) \mathrm{id}\,. \tag{8.13}$$

It follows from the paper [125] that

$$\widetilde{V}_{\mathfrak{B}_k}(\lambda) = V_{-\nu\varepsilon_k} \oplus V_{-(\nu-2)\varepsilon_k} \oplus \ldots \oplus V_{(\nu-2)\varepsilon_k} \oplus V_{\nu\varepsilon_k}, \tag{8.14}$$

where $\nu = m_k - m_{k-1}$ and all summands are one-dimensional weight spaces w.r.t. the Cartan subalgebra $\mathfrak{h}_k$.[1]

Formulas (C.3) and (C.9) imply

$$D^+ :\ V_{j\varepsilon_k} \to V_{(j+2)\varepsilon_k},\ D^- :\ V_{j\varepsilon_k} \to V_{(j-2)\varepsilon_k}\,.$$

The action of operators F_{kk}, D^+, D^- in the space $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ was calculated in [125] w.r.t. some base. In particular, in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ there are no nontrivial invariant subspaces w.r.t. this action.

We shall obtain simpler formulas for the D^+ and D^--action w.r.t. a base in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ with a normalization different from those in [125].

[1] In Appendix C.4 we shall give a proof of expansion (8.14) independent from the theory of Yangians used in [125].

Lemma 8.1. *Let $L_\nu := (-\nu, -\nu+2, \dots, \nu-2, \nu)$. There is a base $(\chi_j)_{j\in L_\nu}$ in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ such that*

$$F_{kk}\chi_j = j\chi_j,\ D^+\chi_j = \frac{1}{4}(j - m_k - m_{k-1} - 2k + 3)(j-\nu)\chi_{j+2}\ , \qquad (8.15)$$

$$D^-\chi_j = \frac{1}{4}(j + m_k + m_{k-1} + 2k - 3)(j+\nu)\chi_{j-2}\ , \qquad (8.16)$$

where $\chi_j = 0$ if $j \notin L_\nu$.

Proof. Since the action of an algebra, generated by operators F_{kk}, D^+, D^-, is irreducible in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$, one can define by induction nonzero elements $\chi_j \in V_{j\varepsilon_k}$, $j \in L_\nu$ such that formulas (8.15) are valid.

Prove by induction formula (8.16). For $j = -\nu$ it is evident. Suppose that (8.16) is valid for $j = -\nu, -\nu+2, \dots, i$, where $i < \nu$. Then using (8.13) one gets

$$\begin{aligned}
&\frac{1}{4}(i - m_k - m_{k-1} - 2k + 3)(i-\nu)D^-\chi_{i+2} = D^-D^+\chi_i\\
&= \left([D^-, D^+] + D^+D^-\right)\chi_i\\
&= \left(\frac{1}{2}F_{kk}^3 - \frac{1}{2}\widetilde{C}F_{kk} - \frac{1}{4}(2k-1)(2k-3)F_{kk}\right)\chi_i\\
&+ \frac{1}{4}(i + m_k + m_{k-1} + 2k - 3)(i+\nu)D^+\chi_{i-2}\\
&= \frac{1}{2}\left(i^3 - i\left(m_k^2 + m_{k-1}^2 + (2k-1)m_k + (2k-3)m_{k-1} + \frac{1}{2}(2k-1)(2k-3)\right)\right)\chi_i\\
&+ \frac{1}{16}(i + m_k + m_{k-1} + 2k - 3)(i+\nu)(i - m_k - m_{k-1} - 2k + 1)(i - 2 - \nu)\chi_i\\
&= \frac{1}{16}(i - m_k - m_{k-1} - 2k + 3)(i-\nu)(i + m_k + m_{k-1} + 2k - 1)(i + 2 + \nu)\chi_i,
\end{aligned}$$

due to the identity

$$\begin{aligned}
&(i - m_k - m_{k-1} - 2k + 3)(i-\nu)(i + m_k + m_{k-1} + 2k - 1)(i + 2 + \nu)\\
&- (i + m_k + m_{k-1} + 2k - 3)(i+\nu)(i - m_k - m_{k-1} - 2k + 1)(i - 2 - \nu)\\
&= 8i^3 - 8i\left(m_k^2 + m_{k-1}^2 + (2k-1)m_k + (2k-3)m_{k-1} + \frac{1}{2}(2k-1)(2k-3)\right)\ .
\end{aligned}$$

Since $(i - m_k - m_{k-1} - 2k + 3)(i - \nu) \neq 0$, we obtain

$$D^-\chi_{i+2} = \frac{1}{4}(i + m_k + m_{k-1} + 2k - 1)(i + 2 + \nu)\chi_i$$

that completes the induction. □

Lemma 8.1, expansion (8.9) and relations (8.10) effectively describe the action of operators D_0, D_1, D_2, D_3 in the space $\mathcal{L}^2\left(\mathrm{SO}(2k+1), \mathrm{SO}(2k-1), d\mu\right)$. Consider the problem of finding all common eigenvectors ψ_D of operators D_0^2, D_1, D_2 and optionally D_3. It is equivalent to the problem of finding all common eigenvectors of operators $D_0^2, D^+ + D^-$ and optionally $D^+ - D^-$ in the space $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$.

Eigenvectors for the operator D_0^2 are

$$c_+\chi_j + c_-\chi_{-j},\ c_\pm \in \mathbb{C},\ j \in L_\nu,\ j \geqslant 0$$

with eigenvalues $-j^2$. Since

$$\begin{aligned}&\left(D^+ + D^-\right)(c_+\chi_j + c_-\chi_{-j})\\ &= \frac{1}{4}(j - m_k - m_{k-1} - 2k + 3)(j-\nu)\left(c_+\chi_{j+2} + c_-\chi_{-j-2}\right)\\ &+\frac{1}{4}(j + m_k + m_{k-1} + 2k - 3)(j+\nu)\left(c_+\chi_{j-2} + c_-\chi_{-j+2}\right),\end{aligned}$$

the requirement

$$\left(D^+ + D^-\right)(c_+\chi_j + c_-\chi_{-j}) \sim c_+\chi_j + c_-\chi_{-j}$$

implies $(j - m_k - m_{k-1} - 2k + 3)(j - \nu) = 0$ that leads to two cases: $j = m_k - m_{k-1}$ and $j = m_k + m_{k-1} + 2k - 3$.

In the first case, one gets

$$\begin{aligned}&\left(D^+ + D^-\right)\left(c_+\chi_{m_k - m_{k-1}} + c_-\chi_{-m_k+m_{k-1}}\right)\\ &= (m_k - m_{k-1})\left(m_k + k - \frac{3}{2}\right)\left(c_+\chi_{m_k-m_{k-1}-2} + c_-\chi_{-m_k+m_{k-1}+2}\right)\end{aligned}$$

that implies one of three possibilities

1. $m_k - m_{k-1} = 0$;
2. $m_k - m_{k-1} - 2 = -m_k + m_{k-1}$;
3. $m_k - m_{k-1} - 2 = 0,\ c_+ + c_- = 0$.

Thus, we obtain the following eigenvectors:

1. $\left(D^+ + D^-\right)\chi_0 = 0$ for $m_k - m_{k-1} = 0$;
2. $\left(D^+ + D^-\right)(\chi_1 + \chi_{-1}) = \left(m_k + k - \frac{3}{2}\right)(\chi_1 + \chi_{-1})$ for $m_k \in \mathbb{N}$, $m_{k-1} = m_k - 1$;
3. $\left(D^+ + D^-\right)(\chi_1 - \chi_{-1}) = -\left(m_k + k - \frac{3}{2}\right)(\chi_1 - \chi_{-1})$ for $m_k \in \mathbb{N}$, $m_{k-1} = m_k - 1$;
4. $\left(D^+ + D^-\right)(\chi_2 - \chi_{-2}) = 0$, $m_{k-1} = m_k - 2$, $m_k = 2, 3, \ldots$.

In the second case one gets $m_k + m_{k-1} + 2k - 3 = j \leqslant m_k - m_{k-1}$ that implies $0 \leqslant m_{k-1} \leqslant \frac{3}{2} - k$ and thus $k = 1$ that contradicts to the assumption $k \geqslant 2$.

Using relations (8.10) this consideration can be summarized in the following proposition.

Proposition 8.3. *For $n = 2k,\ k \geqslant 2$ there are four series of common eigenvectors in $\widetilde{V}_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}),\ m_k, m_{k-1} \in \mathbb{Z}_+$ for the operators D_0^2, D_1, D_2:*

1. $D_0^2\chi_0 = D_3\chi_0 = 0,\ D_1\chi_0 = D_2\chi_0 = -m_k(m_k + 2k - 2)\chi_0,\ m_k = m_{k-1}$;

2. $D_0^2(\chi_1+\chi_{-1}) = -(\chi_1+\chi_{-1})$,
$D_2(\chi_1+\chi_{-1}) = -m_k(m_k+2k-2)(\chi_1+\chi_{-1})$,
$D_1(\chi_1+\chi_{-1}) = \left(-m_k^2-2(k-2)m_k+2k-3\right)(\chi_1+\chi_{-1})$,
$D_3(\chi_1+\chi_{-1}) = \mathbf{i}\left(m_k+k-\frac{3}{2}\right)(\chi_1-\chi_{-1}),\ m_{k-1}=m_k-1, m_k\in\mathbb{N}$

3. $D_0^2(\chi_1-\chi_{-1}) = -(\chi_1-\chi_{-1})$,
$D_1(\chi_1-\chi_{-1}) = -m_k(m_k+2k-2)(\chi_1-\chi_{-1})$,
$D_2(\chi_1-\chi_{-1}) = \left(-m_k^2-2(k-2)m_k+2k-3\right)(\chi_1-\chi_{-1})$,
$D_3(\chi_1-\chi_{-1}) = -\mathbf{i}\left(m_k+k-\frac{3}{2}\right)(\chi_1+\chi_{-1}),\ m_{k-1}=m_k-1, m_k\in\mathbb{N}$;

4. $D_0^2(\chi_2-\chi_{-2}) = -4(\chi_2-\chi_{-2}),\ D_3(\chi_2-\chi_{-2}) = -4\mathbf{i}\left(m_k+k-\frac{3}{2}\right)\chi_0$,
$D_1(\chi_2-\chi_{-2}) = D_2(\chi_2-\chi_{-2}) = \left(-m_k^2-2(k-2)m_k+2k-3\right)(\chi_2-\chi_{-2}), m_{k-1}=m_k-2, m_k=2,3,4,\dots$

Only the first vector is also an eigenvector for the operator D_3.

Multiplicities of corresponding eigenvalues in $\mathcal{L}^2(\mathrm{SO}(n+1),\mathrm{SO}(n-1),d\mu)$ *are equal to* $\dim V_{\mathfrak{B}_k}(m_k\varepsilon_k+m_{k-1}\varepsilon_{k-1})$ *and can be calculated in explicit form using (C.8).*

Consider the case $k=1, n=2$. Now the group K_0 is trivial and therefore $\widetilde{V}_{\mathfrak{B}_1}(\lambda)=V_{\mathfrak{B}_1}(\lambda)$. The algebra $\mathfrak{B}_1=\mathfrak{so}(3,\mathbb{C})\cong\mathfrak{sl}(2,\mathbb{C})=\mathfrak{A}_1$ is spanned by elements F_{11},F_{01},F_{10} with commutator relations

$$[F_{11},F_{01}]=-F_{01},\ [F_{11},F_{10}]=F_{10},\ [F_{10},F_{01}]=F_{11}\ .$$

Its representation theory is well known: all its finite dimensional irreducible modules are of the form

$$V_{\mathfrak{B}_1}(m\varepsilon_1)=V_{-m\varepsilon_1}\oplus V_{-(m-1)\varepsilon_1}\oplus\ldots\oplus V_{(m-1)\varepsilon_1}\oplus V_{m\varepsilon_1}\ ,$$

where $m\in\mathbb{Z}_+\cup\left(\mathbb{Z}_++\frac{1}{2}\right)$, all $V_{j\varepsilon_1}$ are one-dimensional weight subspaces w.r.t. $\mathfrak{h}_1=\mathrm{span}(F_{11})$ and the operators

$$\begin{aligned}F_{10}:\ &V_{j\varepsilon_1}\to V_{(j+1)\varepsilon_1},\ j=-m,\dots,m-1,\\ F_{01}:\ &V_{j\varepsilon_1}\to V_{(j-1)\varepsilon_1}, j=-m+1,\dots,m\end{aligned}$$

are bijective.

We shall consider only $m\in\mathbb{Z}_+$ since

$$\mathcal{L}^2(\mathrm{SO}(3),d\mu)=\bigoplus_{m\in\mathbb{Z}_+}(2m+1)V_{\mathfrak{B}_1}(m\varepsilon_1)\ .$$

Thus, there are additional weight subspaces in the module $V_{\mathfrak{B}_1}(m\varepsilon_1)$ w.r.t. expansion (8.14) and the action of the algebra, generated by the operators $D^+=\frac{1}{2}F_{10}^2, D^-=\frac{1}{2}F_{01}^2$, is not irreducible in $V_{\mathfrak{B}_1}(m\varepsilon_1)$.

One can choose a base $(\chi_j)_{j=-m}^m$ in $V_{\mathfrak{B}_1}(m\varepsilon_1)$ such that

$$\chi_j\in V_{j\varepsilon_1},\ F_{11}\chi_j=j\chi_j,\ F_{10}\chi_j=-\frac{1}{\sqrt{2}}\sqrt{(m-j)(m+j+1)}\chi_{j+1}\ ,$$
$$F_{01}\chi_j=-\frac{1}{\sqrt{2}}\sqrt{(m+j)(m-j+1)}\chi_{j-1}\ ,$$

where as above $\chi_j=0$ for $|j|>m$.

Eigenvectors for the operator $D_0^2 = -F_{11}^2$ are

$$c_+\chi_j + c_-\chi_{-j},\ c_\pm \in \mathbb{C},\ j = 0, 1, \ldots, m$$

with eigenvalues $-j^2$. Since

$$\begin{aligned}&\left(D^+ + D^-\right)(c_+\chi_j + c_-\chi_{-j})\\ &= \frac{1}{4}\sqrt{(m-j)(m+j+1)(m-j-1)(m+j+2)}\,(c_+\chi_{j+2} + c_-\chi_{-j-2})\\ &+ \frac{1}{4}\sqrt{(m+j)(m-j+1)(m+j-1)(m-j+2)}\,(c_+\chi_{j-2} + c_-\chi_{-j+2}),\end{aligned}$$

the requirement

$$\left(D^+ + D^-\right)(c_+\chi_j + c_-\chi_{-j}) \sim c_+\chi_j + c_-\chi_{-j}$$

implies $(m-j)(m+j+1)(m-j-1)(m+j+2) = 0$ that gives two cases: $j = m$ and $j = m-1$.

In the first case, one gets

$$\left(D^+ + D^-\right)(c_+\chi_m + c_-\chi_{-m}) = \frac{1}{2}\sqrt{m(2m-1)}\,(c_+\chi_{m-2} + c_-\chi_{m+2})$$

that implies one of three possibilities

1. $m = j = 0$;
2. $m - 2 = -m$;
3. $m - 2 = 0,\ c_+ + c_- = 0$.

This gives the following eigenvectors:

1. $(D^+ + D^-)\,\chi_0 = 0,\ m = 0$;
2. $(D^+ + D^-)\,(\chi_1 + \chi_{-1}) = \frac{1}{2}\,(\chi_1 + \chi_{-1}),\ m = 1$;
3. $(D^+ + D^-)\,(\chi_1 - \chi_{-1}) = -\frac{1}{2}\,(\chi_1 - \chi_{-1}),\ m = 1$;
4. $(D^+ + D^-)\,(\chi_2 - \chi_{-2}) = 0,\ m = 2$.

It is easily seen that these eigenvectors correspond to eigenvectors from Proposition 8.3 for $m_k = m, m_{k-1} = 0$.

In the second case, it holds

$$\begin{aligned}&\left(D^+ + D^-\right)(c_+\chi_{m-1} + c_-\chi_{-m+1})\\ =&\frac{1}{2}\sqrt{3(2m-1)(m-1)}\,(c_+\chi_{m-3} + c_-\chi_{m+3})\end{aligned}$$

that implies one of three possibilities

1. $m = 1,\ j = 0$;
2. $m - 3 = -m + 1$;
3. $m - 3 = 0,\ c_+ + c_- = 0$.

Thus, one gets the following eigenvectors:

1. $(D^+ + D^-)\,\chi_0 = 0,\ m = 1$;
2. $(D^+ + D^-)\,(\chi_1 + \chi_{-1}) = \frac{3}{2}\,(\chi_1 + \chi_{-1}),\ m = 2$;

3. $(D^+ + D^-)(\chi_1 - \chi_{-1}) = -\frac{3}{2}(\chi_1 - \chi_{-1})$, $m = 2$;
4. $(D^+ + D^-)(\chi_2 - \chi_{-2}) = 0$, $m = 3$.

Now one has

$$D_1^2 = D^+ + D^- + \frac{1}{2}\left(F_{11}^2 - \widetilde{C}\right), \; D_2^2 = -D^+ - D^- + \frac{1}{2}\left(F_{11}^2 - \widetilde{C}\right),$$
$$\{D_1, D_2\} = 2\mathbf{i}\left(D^+ - D^-\right), \; \widetilde{C}\Big|_{\widetilde{V}_{\mathfrak{B}_1}(m\varepsilon_1)} = m(m+1)\,\mathrm{id}\,.$$

Relations (8.10) are valid also in the case $k = 1$ that leads to the following proposition.

Proposition 8.4. *There are eight common eigenvectors in $V_{\mathfrak{B}_1}(m\varepsilon_1)$ for the operators D_0^2, D_1^2, D_2^2:*

1. $D_0^2\chi_0 = D_1^2\chi_0 = D_2^2\chi_0 = \frac{1}{2}\{D_1, D_2\}\chi_0 = 0$, $m = 0$;
2. $D_0^2\chi_0 = \frac{1}{2}\{D_1, D_2\}\chi_0 = 0$, $D_1^2\chi_0 = D_2^2\chi_0 = -\chi_0$, $m = 1$;
3. $D_0^2(\chi_1 + \chi_{-1}) = D_2^2(\chi_1 + \chi_{-1}) = -(\chi_1 + \chi_{-1})$, $D_1^2(\chi_1 + \chi_{-1}) = 0$,
   $\frac{1}{2}\{D_1, D_2\}(\chi_1 + \chi_{-1}) = \frac{\mathbf{i}}{2}(\chi_1 - \chi_{-1})$, $m = 1$;
4. $D_0^2(\chi_1 - \chi_{-1}) = D_1^2(\chi_1 - \chi_{-1}) = -(\chi_1 - \chi_{-1})$, $D_2^2(\chi_1 - \chi_{-1}) = 0$,
   $\frac{1}{2}\{D_1, D_2\}(\chi_1 - \chi_{-1}) = -\frac{\mathbf{i}}{2}(\chi_1 + \chi_{-1})$, $m = 1$;
5. $D_0^2(\chi_2 - \chi_{-2}) = -4(\chi_2 - \chi_{-2})$,
   $D_1^2(\chi_2 - \chi_{-2}) = D_2^2(\chi_2 - \chi_{-2}) = -(\chi_2 - \chi_{-2})$,
   $\frac{1}{2}\{D_1, D_2\}(\chi_2 - \chi_{-2}) = -\sqrt{6}\mathbf{i}\chi_0$, $m = 2$;
6. $D_0^2(\chi_1 + \chi_{-1}) = D_1^2(\chi_1 + \chi_{-1}) = -(\chi_1 + \chi_{-1})$,
   $D_2^2(\chi_1 + \chi_{-1}) = -4(\chi_1 + \chi_{-1})$,
   $\frac{1}{2}\{D_1, D_2\}(\chi_1 + \chi_{-1}) = \frac{3}{2}\mathbf{i}(\chi_1 - \chi_{-1})$, $m = 2$;
7. $D_0^2(\chi_1 - \chi_{-1}) = D_2^2(\chi_1 - \chi_{-1}) = -(\chi_1 - \chi_{-1})$,
   $D_1^2(\chi_1 - \chi_{-1}) = -4(\chi_1 - \chi_{-1})$,
   $\frac{1}{2}\{D_1, D_2\}(\chi_1 - \chi_{-1}) = -\frac{3}{2}\mathbf{i}(\chi_1 + \chi_{-1})$, $m = 2$;
8. $D_0^2(\chi_2 - \chi_{-2}) = D_1^2(\chi_2 - \chi_{-2}) = D_2^2(\chi_2 - \chi_{-2}) = -4(\chi_2 - \chi_{-2})$,
   $\frac{1}{2}\{D_1, D_2\}(\chi_2 - \chi_{-2}) = -\sqrt{30}\mathbf{i}\chi_0$, $m = 3$.

Only the first and the second vectors are also eigenvectors for the operator $\frac{1}{2}\{D_1, D_2\}$.

Multiplicities of corresponding eigenvalues in $\mathcal{L}^2(\mathrm{SO}(3), d\mu)$ are $2m+1$.

8.2.2 The Case $n = 2k - 1$

Here we use notations from appendix C.2. The algebra $\mathfrak{D}_k$ is considered there as a subalgebra of $\mathfrak{B}_k$. Therefore, one can easily obtain analogs for formulas (8.7) simply by deleting the terms F_{k0} and F_{0k}:

$$D_0 = -\mathbf{i}F_{kk}, \; D_1 = \frac{1}{2}\sum_{j=1}^{k-1}\{F_{-kj} + F_{kj}, F_{k,-j} + F_{-k,-j}\},$$

$$D_2 = \frac{1}{2}\sum_{j=1}^{k-1}\{F_{-kj} - F_{kj}, F_{k,-j} - F_{-k,-j}\}\,,$$

$$D_3 = \mathbf{i}\sum_{j=1}^{k-1}(F_{kj}F_{k,-j} - F_{-kj}F_{-k,-j})\ .$$

Let $\mathcal{R}_{\ell,i}$ be the space $V_{\mathfrak{D}_k}(\lambda)$ for a highest weight (C.10), where $m_i \in \mathbb{Z}_+,\ i \geqslant 2, m_1 \in \mathbb{Z}$, and $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ be a subspace of $V_{\mathfrak{D}_k}(\lambda)$ annulled by the subalgebra $\mathfrak{k}_0^{\mathbb{C}} \cong \mathfrak{D}_{k-1}$. Reasoning as above in the case $n = 2k$, one gets that $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ is nontrivial iff

$$\lambda = m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1},\ m_k \geqslant |m_{k-1}|,\ m_k \in \mathbb{Z}_+,\ m_{k-1} \in \mathbb{Z}'_k, \tag{8.17}$$

where $\mathbb{Z}'_k = \mathbb{Z}_+$ for $k \geqslant 3$ and $\mathbb{Z}'_2 = \mathbb{Z}$. In this case, it holds

$$\dim \widetilde{V}_{\mathfrak{D}_k}(\lambda) = m_k - |m_{k-1}| + 1\ . \tag{8.18}$$

Below in the present subsection we suppose that condition (8.17) is valid. This leads to the expansion

$$\mathcal{L}^2\left(\mathrm{SO}(2k), \mathrm{SO}(2k-2), d\mu\right) = \bigoplus_{\substack{m_k \geqslant |m_{k-1}| \\ m_k \in \mathbb{Z}_+,\ m_{k-1} \in \mathbb{Z}'_k}} (m_k - |m_{k-1}| + 1)V_{\mathfrak{D}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right)$$

of the left $\mathrm{SO}(2k)$-space $\mathcal{L}^2\left(\mathrm{SO}(2k), \mathrm{SO}(2k-2), d\mu\right)$ and to the expansion

$$\mathcal{L}^2\left(\mathrm{SO}(2k), \mathrm{SO}(2k-2), d\mu\right) = \bigoplus_{\substack{m_k \geqslant |m_{k-1}| \\ m_k \in \mathbb{Z}_+,\ m_{k-1} \in \mathbb{Z}'_k}} \left(\dim V_{\mathfrak{D}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right)\right)\widetilde{V}_{\mathfrak{D}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right), \tag{8.19}$$

of the same space as a $\mathrm{Diff}_{\mathrm{SO}(2k)}(\mathrm{SO}(2k)/\,\mathrm{SO}(2k-2))$-module, where the dimension $\dim V_{\mathfrak{D}_k}\left(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}\right)$ is given by (C.8). Now let

$$D^+ := \sum_{j=1}^{k-1} F_{kj}F_{k,-j}\ ,\ D^- := \sum_{j=1}^{k-1} F_{-kj}F_{-k,-j};\,,$$

$$\widetilde{C} := C|_{\mathcal{L}^2(\mathrm{SO}(2k),\mathrm{SO}(2k-2),d\mu)} = F_{kk}^2 + \sum_{j=1}^{k-1}\left(\{F_{kj}, F_{jk}\} + \{F_{k,-j}, F_{-jk}\}\right)$$

be operators from $\mathcal{L}^2\left(\mathrm{SO}(2k), \mathrm{SO}(2k-2), d\mu\right)$, where C is the universal Casimir operator (C.11).

Formulas (8.10) and (8.11) are valid without any modification and formula (8.12) becomes

$$[D^+, D^-] = -\frac{1}{2}F_{kk}^3 + \frac{1}{2}\widetilde{C}F_{kk} + (k-1)(k-2)F_{kk} \,.$$

Now

$$\widetilde{C}\Big|_{\widetilde{V}_{\mathfrak{D}_k}(\lambda)} = \Big((m_k + k - 1)^2 + (m_{k-1} + k - 2)^2 - (k-1)^2 - (k-2)^2\Big)\,\mathrm{id} \;.$$

From [124] it follows[2] that

$$\widetilde{V}_{\mathfrak{D}_k}(\lambda) = V_{-\nu\varepsilon_k} \oplus V_{-(\nu-2)\varepsilon_k} \oplus \ldots \oplus V_{(\nu-2)\varepsilon_k} \oplus V_{\nu\varepsilon_k}, \tag{8.20}$$

where $\nu = m_k - |m_{k-1}|$, all summands are one-dimensional weight spaces w.r.t. the Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{D}_k$ and the algebra, generated by the operators D^+, D^-, acts in $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ in an irreducible way.

Again we shall simplify formulas for this action w.r.t. [124] using another base. The next lemma can be proved completely similar to the proof of Lemma 8.1.

Lemma 8.2. *Let $\nu := m_k - |m_{k-1}|$, $L_\nu := (-\nu, -\nu+2, \ldots, \nu-2, \nu)$. There is a base $(\chi_j)_{j\in L_\nu}$ in $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ such that*

$$F_{kk}\chi_j = j\chi_j,\; D^+\chi_j = \frac{1}{4}(j - m_k - |m_{k-1}| - 2k + 4)(j-\nu)\chi_{j+2},$$
$$D^-\chi_j = \frac{1}{4}(j + m_k + |m_{k-1}| + 2k - 4)(j+\nu)\chi_{j-2},$$

where $\chi_j = 0$ if $j \notin L_\nu$.

Arguing as in the $\mathfrak{B}_k$-case one gets the following proposition.

Proposition 8.5. *For $n = 2k-1$, $k \geqslant 2$, there are four series of common eigenvectors in $\widetilde{V}_{\mathfrak{D}_k}(m_k\varepsilon_k + |m_{k-1}|\varepsilon_{k-1})$, $m_k \in \mathbb{Z}_+$, $m_{k-1} \in \mathbb{Z}'_k$ for the operators D_0^2, D_1, D_2:*

1. $D_0^2\chi_0 = D_3\chi_0 = 0$, $D_1\chi_0 = D_2\chi_0 = -m_k(m_k + 2k - 3)\chi_0$, $m_k = |m_{k-1}|$;
2. $D_0^2(\chi_1+\chi_{-1}) = -(\chi_1+\chi_{-1})$, $D_2(\chi_1+\chi_{-1}) = -m_k(m_k+2k-3)(\chi_1+\chi_{-1})$,
   $D_1(\chi_1+\chi_{-1}) = \left(-m_k^2 + (5-2k)m_k + 2k - 4\right)(\chi_1+\chi_{-1})$,
   $D_3(\chi_1+\chi_{-1}) = \mathbf{i}\,(m_k + k - 2)\,(\chi_1 - \chi_{-1})$, $|m_{k-1}| = m_k - 1, m_k \in \mathbb{N}$
3. $D_0^2(\chi_1-\chi_{-1}) = -(\chi_1-\chi_{-1})$, $D_1(\chi_1-\chi_{-1}) = -m_k(m_k+2k-3)(\chi_1-\chi_{-1})$,
   $D_2(\chi_1-\chi_{-1}) = \left(-m_k^2 + (5-2k)m_k + 2k - 4\right)(\chi_1-\chi_{-1})$,
   $D_3(\chi_1-\chi_{-1}) = -\mathbf{i}\,(m_k + k - 2)\,(\chi_1 + \chi_{-1})$, $|m_{k-1}| = m_k - 1, m_k \in \mathbb{N}$;
4. $D_0^2(\chi_2-\chi_{-2}) = -4(\chi_2-\chi_{-2})$, $D_3(\chi_2-\chi_{-2}) = -4\mathbf{i}\,(m_k + k - 2)\,\chi_0$,
   $D_1(\chi_2-\chi_{-2}) = D_2(\chi_2-\chi_{-2}) = \left(-m_k^2 + (5-2k)m_k + 2k - 4\right)(\chi_2-\chi_{-2})$,
   $|m_{k-1}| = m_k - 2, m_k = 2, 3, 4, \ldots$

Only the first vector is also an eigenvector for the operator D_3.

[2] See also appendix C.4 for a proof independent from [124].

Multiplicities of corresponding eigenvalues in $\mathcal{L}^2\,(\mathrm{SO}(n+1), \mathrm{SO}(n-1), d\mu)$ *are equal to* $\dim V_{\mathfrak{D}_k}\,(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})$ *and can be calculated in explicit form using (C.8).*

Remark 8.1. *For* $k = 2$ *a value of* $m_{k-1} = m_1$ *can has an arbitrary sign and one gets eight common eigenvectors found in [184].*

Remark 8.2. *Results of Propositions 8.3, 8.4 and 8.5 correspond to Proposition 8.2 and are even more restrictive. Indeed, if*

$$\psi_D \in \mathcal{L}^2(\mathrm{SO}(n+1), \mathrm{SO}(n-1), d\mu)$$

is an eigenfunction for operators D_0^2, D_1, D_2 *and* D_3 *for* $n \geqslant 3$*, then* $D_0\psi_D = D_3\psi_D = 0$*,* $D_1\psi_D = D_2\psi_D$*. Similarly, if* $\psi_D \in \mathcal{L}^2\,(\mathrm{SO}(3), d\mu)$ *is an eigenfunction for operators* D_0^2, D_1^2, D_2^2 *and* $\{D_1, D_2\}$*, then* $D_0\psi_D = \{D_1, D_2\}\psi_D = 0$*,* $D_1^2\psi_D = D_2^2\psi_D$*.*

Remark 8.3. *Common eigenfunctions in Propositions 8.3, 8.4 and 8.5 correspond to the case* $Q = \mathbf{S}^n$*. For the projective space* $\mathbf{P}^n(\mathbb{R})$ *one should consider the subspace of the space* $\mathcal{L}^2\,(\mathrm{SO}(n+1), \mathrm{SO}(n-1), d\mu)$*, consisting of functions, invariant w.r.t. the operator* $\exp(\pi D_0)$*, since it corresponds to the antipodal transformation on* $\mathbf{S}^n$*. Evidently, an eigenfunction for the operator* D_0 *is invariant w.r.t.* $\exp(\pi D_0)$ *iff it corresponds to an eigenvalue of the form* $2\mathbf{i}\ell$*,* $\ell \in \mathbb{Z}$*. Therefore in the case* $Q = \mathbf{P}^n(\mathbb{R})$ *one should restrict oneself with cases 1,4 of Propositions 8.3, 8.5 and cases 1,2,5,8 of Proposition 8.4.*

8.3 Scalar Spectral Equations and Some Energy Levels for the Two-Body Problem

For an arbitrary compact two-point homogeneous space one gets from (5.22)–(5.25) and (8.1) the spectral problem for the eigenfunction $\psi = f(r)$:

$$-\frac{(1+r^2)^2}{8mR^2}f'' - \frac{1+r^2}{8mR^2 r}\left(q_1 + q_2 + (2-q_2)r^2\right)f' + (V(r) - E)f = 0,\ 0 < r < \infty, \tag{8.21}$$

where multiplicities q_1 and q_2 are given in Proposition 1.2. In this section, m denotes reduced mass (5.19) and integers m_k correspond to highest weights in $\mathfrak{so}(n+1)$-modules.

Consider the case $Q = \mathbf{S}^n$. Let D_i', $i = 1, 2, 3$ be given as in Proposition 8.2 and $D_0^2\psi_D = \delta_0\psi_D$, $D_i'\psi_D = \delta_i\psi_D$, $i = 1, 2$.

In accordance with Remark 8.2 there are two main cases:

1. $D_3'\psi_D = 0$, $\delta_0 = 0, \delta_1 = \delta_2$, particle masses are arbitrary;
2. $D_3'\psi_D \not\sim \psi_D$, particle masses are equal.

In the first case,

$$(CD_1' + AD_2' + 2BD_3')\,\psi_D = \delta_1(C + A)\psi_D = \frac{(1+r^2)^2}{4mR^2r^2}\delta_1\psi_D$$

and in the second case,

$$A = \frac{1+r^2}{4mR^2r^2},\; B \equiv 0,\; C = \frac{1+r^2}{4mR^2}\;.$$

In all cases, the spectral (8.1) has the form

$$f'' + \frac{n-1+(3-n)r^2}{(1+r^2)r} f' + \frac{8}{(1+r^2)^2}\left(mR^2(E-V(r)) - \frac{a}{r^2} - b - cr^2\right) f = 0,$$
$$a, b, c \geqslant 0,\; 0 < r < \infty. \tag{8.22}$$

where coefficients a, b, c are described below.

For eigenfunctions ψ_D classified in Proposition 8.3 ($n = 2k, k = 2, 3, \ldots$) one has

1. $a = c = m_k(m_k + 2k - 2)/8$, $b = 2a$, $m_k \in \mathbb{Z}_+$, masses are arbitrary;
2. $a = m_k(m_k + 2k - 2)/8$, $b = (m_k^2 + (2k-3)m_k - k + 2)/4$, $c = (m_k^2 + 2(k-2)m_k - 2k + 3)/8$, $m_k \in \mathbb{N}$, masses are equal;
3. $a = (m_k^2 + 2(k-2)m_k - 2k + 3)/8$, $b = (m_k^2 + (2k-3)m_k - k + 2)/4$, $c = m_k(m_k + 2k - 2)/8$, $m_k \in \mathbb{N}$, masses are equal;
4. $a = c = (m_k^2 + 2(k-2)m_k - 2k + 3)/8$, $b = (m_k^2 + 2(k-2)m_k - 2k + 5)/4$, $m_k = 2, 3, \ldots$, masses are equal.

Proposition 8.4 ($n = 2$) gives the following values for a, b, c:

1. $a = c = b = 0$, masses are arbitrary;
2. $a = c = 1/8, b = 1/4$, masses are arbitrary;
3. $a = 1/8, b = 1/4, c = 0$, masses are equal;
4. $a = 0, b = 1/4, c = 1/8$, masses are equal;
5. $a = c = 1/8, b = 3/4$, masses are equal;
6. $a = 1/2, b = 3/4, c = 1/8$, masses are equal;
7. $a = 1/8, b = 3/4, c = 1/2$, masses are equal;
8. $a = c = 1/2, b = 3/2$, masses are equal.

Finally, Proposition 8.5 corresponds to the following cases ($n = 2k-1, k = 2, 3, \ldots$):

1. $a = c = m_k(m_k + 2k - 3)/8$, $b = 2a$, $m_k \in \mathbb{Z}_+$, masses are arbitrary;
2. $a = m_k(m_k + 2k - 3)/8$, $b = (m_k^2 + (2k-4)m_k - k + \frac{5}{2})/4$, $c = (m_k^2 + (2k-5)m_k - 2k + 4)/8$, $m_k \in \mathbb{N}$, masses are equal;
3. $a = (m_k^2 + (2k-5)m_k - 2k + 4)/8$, $b = (m_k^2 + (2k-4)m_k - k + \frac{5}{2})/4$, $c = m_k(m_k + 2k - 3)/8$, $m_k \in \mathbb{N}$, masses are equal;
4. $a = c = (m_k^2 + (2k-5)m_k - 2k + 4)/8$, $b = (m_k^2 + (2k-5)m_k - 2k + 6)/4$, $m_k = 2, 3, \ldots$, masses are equal.

One can write (8.21) and (8.22) in the general form:

$$f'' + \frac{q_1+q_2+(2-q_2)r^2}{(1+r^2)r} f' + \frac{8}{(1+r^2)^2}\left(mR^2(E-V(r)) - \frac{a}{r^2} - b - cr^2\right) f = 0,$$
$$a, b, c \geqslant 0,\; 0 < r < \infty\;. \tag{8.23}$$

We shall consider (8.23) for the Coulomb and oscillator potentials.

8.3.1 Coulomb Potential

For the Coulomb potential,

$$V_c = \frac{\gamma}{2R}\left(r - \frac{1}{r}\right), \ \gamma > 0 \tag{8.24}$$

Theorems 2.11 and 5.1 imply the self-adjointness of the two-body Hamiltonian[3] with its domain defined by (2.28), where $V_1 = 0$ for $0 < r < 1$ and $V_1 = V_c$ for $1 \leqslant r < \infty$.

Equation (8.23) for $V = V_c$ is the Fuchsian differential equation with four singular points $r = 0, \pm\mathbf{i}, \infty$ and corresponding characteristic exponents:

$$\begin{aligned}
\rho_\pm^{(0)} &= \frac{1}{2}\left(1 - q_1 - q_2 \pm \sqrt{(q_1+q_2-1)^2 + 32a}\right), \\
\rho_\pm^{(\infty)} &= \frac{1}{2}\left(1 - q_2 \pm \sqrt{(q_2-1)^2 + 32c}\right), \\
\rho_\pm^{(\mathbf{i})} &= \frac{1}{2}\left(\frac{1}{2}q_1 + q_2 \pm \sqrt{\left(\frac{1}{2}q_1 + q_2\right)^2 + 8\left(mER^2 - \mathbf{i}mR\gamma + a - b + c\right)}\right), \\
\rho_\pm^{(-\mathbf{i})} &= \frac{1}{2}\left(\frac{1}{2}q_1 + q_2 \pm \sqrt{\left(\frac{1}{2}q_1 + q_2\right)^2 + 8\left(mER^2 + \mathbf{i}mR\gamma + a - b + c\right)}\right).
\end{aligned} \tag{8.25}$$

The same arguments as for the one-body Coulomb problem in Sect. 6.3 show that the function $f(r),\ r \in (0,\infty)$ should be $\sim r^{\rho_+^{(0)}}$ as $r \to +0$. Consider its asymptotics as $r \to +\infty$. Let the function ψ_D be as in (8.1). Due to Theorem 5.1 the measure $d\nu$ on the interval $(0,+\infty) \ni r$ is

$$\frac{r^{q_1+q_2}dr}{(1+r^2)^{1+\frac{1}{2}q_1+q_2}} \sim \frac{dr}{r^{q_2+2}}, \ r \to +\infty .$$

Let $\zeta = 1/r$ be the local coordinate near the point $r = +\infty$, then $d\nu \sim \zeta^{q_2} d\zeta$ as $\zeta \to 0$ and the asymptotic $\sim \zeta^{1-q_2}$ corresponds to a solution of the equation

$$\triangle\varphi = \text{const}\cdot\delta(\zeta), \ \text{const} \neq 0 .$$

From $\rho_-^{(\infty)} \leqslant 1 - q_2$ it follows that if $f(r) \sim r^{-\rho_-^{(\infty)}}$ as $r \to +\infty$, then the function $H(f(r)\psi_D)$ is not in $\mathcal{L}^2(Q \times Q, d\mu_2)$, where μ_2 is the measure on $Q \times Q$, corresponding to metric (5.1). Thus, it should be $f(r) \sim r^{-\rho_+^{(\infty)}}$ as $r \to +\infty$ (note that $\rho_+^{(\infty)} \geqslant 0$).

Consider the problem of reducing of (8.23) with potential (8.24) to the hypergeometric equation via reducing (8.23) to the Heun equation by transformations (B.3), (B.4) and then using Theorem B.1.

Singular points of (8.23) form a harmonic quadruple (see appendix B). Therefore, one can use only the first case of Theorem B.1. Move singular points

[3] It is the Schrödinger operator on the space $M = Q \times Q$ w.r.t. metric (5.1).

$(0, \pm\mathbf{i}, \infty)$ of equation (8.23) to the quadruple $(0, 1, 2, \infty)$ by a fractional linear transformation $t = \mu(r)$ of independent variable.

Since the order of singular points on a circle or on a line is conserved by such transformation, only two possibilities can occur. The first one corresponds to the map of the unordered pair $(\pm\mathbf{i})$ into the unordered pair $(0, 2)$. The second one corresponds to the map of the unordered pair $(0, \infty)$ into the unordered pair $(0, 2)$.

Then, one can reduce the transformed equation to the Heun one by a substitution of the form (B.4). One of requirements of the first case of Theorem B.1 is the equality of characteristic exponents at points 0 and 2. In terms of characteristic exponents (8.25) it means that either $|\rho_+^{(\mathbf{i})} - \rho_-^{(\mathbf{i})}| = |\rho_+^{(-\mathbf{i})} - \rho_-^{(-\mathbf{i})}|$ or $|\rho_+^{(0)} - \rho_-^{(0)}| = |\rho_+^{(\infty)} - \rho_-^{(\infty)}|$. The first possibility can not occur for a nontrivial γ. Therefore, without loss of generality, one can consider the map

$$t = \mu(r) := \frac{2r}{r + \mathbf{i}}, \quad \mu : \ (-\mathbf{i}, 0, \mathbf{i}, \infty) \to (\infty, 0, 1, 2) \ .$$

This map transforms (8.23) with potential (8.24) into the equation

$$f_{tt}(t) + \mathcal{A}(t) f_t(t) - \mathcal{B}(t) f(t) = 0, \ |t - 1| = 1, \ \operatorname{Im} t < 0, \tag{8.26}$$

where

$$\mathcal{A}(t) = \frac{(q_1 + 2q_2 + 2)t^2 - 4(q_1 + q_2 + 1)t + 4(q_1 + q_2)}{2t(t-1)(t-2)},$$

$$\mathcal{B}(t) = 2\frac{m\big(ER^2t^2(t-2)^2 + R\gamma\mathbf{i}t(t-2)(t^2 - 2t + 2)\big) + a(t-2)^4 - bt^2(t-2)^2 + ct^4}{t^2(t-1)^2(t-2)^2}.$$

The substitution

$$f(t) = t^{\rho_+^{(0)}} (t-1)^{\rho_+^{(\mathbf{i})}} (t-2)^{\rho_+^{(\infty)}} w(t)$$

transforms (8.26) into Heun equation (B.14) with the parameter γ' instead of γ, where

$$\alpha = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \rho_+^{(-\mathbf{i})}, \ \beta = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \rho_-^{(-\mathbf{i})}, \ d = 2,$$
$$\gamma' = 1 - \rho_-^{(0)} + \rho_+^{(0)}, \ \delta = 1 - \rho_-^{(\mathbf{i})} + \rho_+^{(\mathbf{i})}, \ \varepsilon = 1 - \rho_-^{(\infty)} + \rho_+^{(\infty)}.$$

Here, $t^{\rho_+^{(0)}} (t-1)^{\rho_+^{(\mathbf{i})}} (t-2)^{\rho_+^{(\infty)}}$ means the function holomorphic on $\mathbb{C}\backslash(-\infty, 2]$ and real for real $t > 2$. Restrictions on asymptotics of the function f near the points $r = 0, \infty$ are equivalent to the boundedness of the function $w(t)$ near the points $t = 0, 2$.

Obviously, the accessory parameter q can be found as

$$q = -2\lim_{t\to 0} t\left(-\mathcal{B}(t) + \left(\frac{\rho_+^{(0)}}{t} + \frac{\rho_+^{(\mathbf{i})}}{t-1} + \frac{\rho_+^{(\infty)}}{t-2}\right)\mathcal{A}(t) + \frac{\rho_+^{(0)}(\rho_+^{(0)}-1)}{t^2} + \frac{2\rho_+^{(0)}\rho_+^{(\mathbf{i})}}{t(t-1)}\right.$$
$$\left. + \frac{2\rho_+^{(0)}\rho_+^{(\infty)}}{t(t-2)}\right) = 4\rho_+^{(0)}\rho_+^{(\mathbf{i})} + 2\rho_+^{(0)}\rho_+^{(\infty)} - (q_1+q_2-2)\rho_+^{(0)} + (q_1+q_2)(2\rho_+^{(\mathbf{i})} + \rho_+^{(\infty)})$$
$$- 4mR\gamma\mathbf{i} + 16a\ . \tag{8.27}$$

Theorem B.1 implies that this Heun equation can be transformed into the hypergeometric equation by a rational change of independent variable $t \to z:\ z = P(t)$, where P is a rational function, iff

$$\gamma' = \varepsilon, \tag{8.28}$$
$$\alpha\beta - q = 0\ . \tag{8.29}$$

Equation (8.28) can be rewritten as

$$(q_1+q_2-1)^2 + 32a = (q_2-1)^2 + 32c$$

or as

$$16(c-a) = q_1(\frac{1}{2}q_1 + q_2 - 1)\ . \tag{8.30}$$

Using the equalities

$$\alpha = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)}$$
$$+ \frac{1}{2}\left(\frac{1}{2}q_1 + q_2 + \sqrt{\left(\frac{1}{2}q_1+q_2\right)^2 + 8\left(mER^2 + \mathbf{i}mR\gamma + a - b + c\right)}\right),$$
$$\beta = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)}$$
$$+ \frac{1}{2}\left(\frac{1}{2}q_1 + q_2 - \sqrt{\left(\frac{1}{2}q_1+q_2\right)^2 + 8\left(mER^2 + \mathbf{i}mR\gamma + a - b + c\right)}\right),$$

one can rewrite (8.29) as

$$\left(\rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \frac{1}{2}(\frac{1}{2}q_1 + q_2)\right)^2$$
$$-\frac{1}{4}\left((\frac{1}{2}q_1+q_2)^2 + 8\left(mER^2 + \mathbf{i}mR\gamma + a - b + c\right)\right)$$
$$-4\rho_+^{(0)}\rho_+^{(\mathbf{i})} - 2\rho_+^{(0)}\rho_+^{(\infty)} + (q_1+q_2-2)\rho_+^{(0)} \tag{8.31}$$
$$-(q_1+q_2)(2\rho_+^{(\mathbf{i})} + \rho_+^{(\infty)}) + 4mR\gamma\mathbf{i} - 16a$$
$$= \left(\rho_+^{(0)}\right)^2 + \left(\rho_+^{(\mathbf{i})}\right)^2 + \left(\rho_+^{(\infty)}\right)^2 + 2\rho_+^{(\mathbf{i})}\left(\rho_+^{(\infty)} - \rho_+^{(0)}\right) + \left(\frac{3}{2}q_1 + 2q_2 - 2\right)\rho_+^{(0)}$$
$$- \left(\frac{3}{2}q_1 + q_2\right)\rho_+^{(\mathbf{i})} - \frac{1}{2}q_1\rho_+^{(\infty)} + 2mR\gamma\mathbf{i} - 2mER^2 - 18a + 2b - 2c = 0\ .$$

Excluding squares of values $\rho_+^{(0)}$, $\rho_+^{(\mathbf{i})}$, $\rho_+^{(\infty)}$ from (8.32) with the help of obvious equations

$$\left(\rho_+^{(0)}\right)^2 + (q_1 + q_2 - 1)\rho_+^{(0)} - 8a = 0,$$
$$\left(\rho_+^{(\mathbf{i})}\right)^2 - (\frac{1}{2}q_1 + q_2)\rho_+^{(\mathbf{i})} - 2mR\,(RE - \gamma\mathbf{i}) - 2(a - b + c) = 0,$$
$$\left(\rho_+^{(\infty)}\right)^2 + (q_2 - 1)\rho_+^{(\infty)} - 8c = 0$$

for characteristic exponents, one gets

$$\left(2\rho_+^{(\mathbf{i})} - \frac{1}{2}q_1 - q_2 + 1\right)\left(\rho_+^{(\infty)} - \rho_+^{(0)}\right) - q_1\rho_+^{(\mathbf{i})} + 8(c - a) = 0\,. \qquad (8.32)$$

It follows from (8.28) that $\rho_+^{(\infty)} - \rho_+^{(0)} = \frac{1}{2}q_1$ and then (8.32) leads to

$$\frac{1}{2}q_1\left(1 - q_2 - \frac{1}{2}q_1\right) + 8(c - a) = 0$$

modulo (8.28), which is equivalent to (8.30) and thus to (8.28) itself.

Hence we came to the conclusion that (8.29) *is a consequence of* (8.28). The condition (8.28) is valid, particularly, in the case $q_1 = 0, a = c$, when $\rho_\pm^{(0)} = \rho_\pm^{(\infty)}$. This case corresponds to the real spheres $\mathbf{S}^n$ with $q_2 = n - 1$, i.e., to cases 1,4 of Propositions 8.3, 8.5 and cases 1,2,5,8 of Proposition 8.4.[4]

From now till the end of the present section we suppose that condition (8.28) is valid.

The fist case of Theorem B.1 implies then that the function w w.r.t. a new independent variable[5]

$$z := 1 - (t - 1)^2 = t(2 - t) \qquad (8.33)$$

satisfies the hypergeometric equation:

$$z(1 - z)w''(z) + (\widetilde{\gamma} - (\widetilde{\alpha} + \widetilde{\beta} + 1)z)w'(z) - \widetilde{\alpha}\widetilde{\beta}w(z) = 0 \qquad (8.34)$$

with the P-symbol

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \widetilde{\alpha} & ;z \\ 1 - \widetilde{\gamma} & \widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta} & \widetilde{\beta} & \end{Bmatrix}.$$

The correspondence between characteristic exponents of the Heun and the hypergeometric equations connected by (8.33) implies

$$\widetilde{\alpha} = \frac{1}{2}\alpha,\ \widetilde{\beta} = \frac{1}{2}\beta,\ \widetilde{\gamma} = \gamma'\,.$$

[4] The projective spaces $\mathbf{P}^n(\mathbb{R})$ are beyond the problem under consideration since potential (8.24) is not projectible from $\mathbf{S}^n$ onto $\mathbf{P}^n(\mathbb{R}) \cong \mathbf{S}^n/\mathrm{Z}_2$

[5] The variable z corresponds to the variable z in (6.46).

Since

$$z - 1 = -\left(\frac{r-i}{r+i}\right)^2,$$

the half-line $[0, \infty]$ on the r-plane is mapped into the circle on the z-plane defined by the equation $|z-1| = 1$, while the values $r = 0, \infty$ correspond to the point $z = 0$. Therefore, the situation is the same as for the one-particle Coulomb problem in Sect. 6.3.2 up to notations.[6]

Indeed, it holds

$$\begin{aligned}
\widetilde{\gamma} &= 1 + \sqrt{(q_2-1)^2 + 32a} \in \mathbb{R},\ 1 - \widetilde{\gamma} \leqslant 0, \\
\widetilde{\alpha} &= \frac{1}{2} + \frac{1}{2}\sqrt{(q_2-1)^2 + 32a} + \frac{1}{4}(s + \bar{s}) \in \mathbb{R}, \\
\widetilde{\beta} &= \frac{1}{2} + \frac{1}{2}\sqrt{(q_2-1)^2 + 32a} + \frac{1}{4}(-s + \bar{s}) \notin \mathbb{R},
\end{aligned}$$

where $s := \sqrt{\left(\frac{1}{2}q_1 + q_2\right)^2 + 8\left(mER^2 + \mathbf{i}mR\gamma + 2a - b\right)}$.

Thus, similar to the Coulomb problem in Sect. 6.3.2 one comes to one of the following two equalities $\widetilde{\alpha} = -k+1$, $k \in \mathbb{N}$ or $\widetilde{\gamma} - \widetilde{\alpha} = -k+1$, $k \in \mathbb{N}$. Without loss of generality suppose that $\operatorname{Re} s < 0$. Then the second equality is impossible and the first equality yields

$$s = 1 - 2k - \sqrt{(q_2-1)^2 + 32a} + \frac{4\mathbf{i}mR\gamma}{1 - 2k - \sqrt{(q_2-1)^2 + 32a}}$$

and finally, due to the definition of s, one gets

$$\begin{aligned}
E_k = &\frac{1}{8mR^2}\Big(2(2k-1)\sqrt{(q_2-1)^2 + 32a} + 4k^2 - 4k + 2 - \frac{1}{2}q_1q_2 \\
&-\frac{1}{2}q_1 - 2q_2 + 16a + 8b\Big) - \frac{2m\gamma^2}{\left(\sqrt{(q_2-1)^2 + 32a} + 2k - 1\right)^2},\ k \in \mathbb{N}\,.
\end{aligned}$$

Particularly, for the real spheres $\mathbf{S}^n$ in the case $q_1 = 0$, $q_2 = n-1$, $a = c$, which corresponds to cases 1,4 of Propositions 8.3, 8.5 and cases 1,2,5,8 of Proposition 8.4, one gets

$$\begin{aligned}
E_k = &\frac{1}{mR^2}\left(\frac{1}{2}(k^2 - k + 1) - \frac{n}{4} + 2a + b + \frac{2k-1}{4}\sqrt{(n-2)^2 + 32a}\right) \\
&- \frac{2m\gamma^2}{\left(\sqrt{(n-2)^2 + 32a} + 2k - 1\right)^2},\ k \in \mathbb{N}\,.
\end{aligned}$$

Taking into account all transformations used while reducing (8.23) to the hypergeometric equation, we get the following expression for the corresponding radial eigenfunctions (up to an arbitrary constant nonzero factor)

$$f_k(r) = \frac{r^{\rho_+^{(0)}}(r - \mathbf{i})^{\rho_+^{(\mathbf{i})}}}{(r + \mathbf{i})^{\rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)}}} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!(k-j-1)!} \frac{(\widetilde{\beta})_j}{(\widetilde{\gamma})_j} \frac{(4r\mathbf{i})^j}{(r+\mathbf{i})^{2j}},$$

where $\rho_+^{(0)}$, $\rho_+^{(\mathbf{i})}$, $\rho_+^{(\infty)}$, $\widetilde{\beta}$ and $\widetilde{\gamma}$ are given by above formulas for $E = E_k$.

[6] Particularly $\widetilde{\alpha}$ instead of α and $\widetilde{\beta}$ instead of β.

8.3.2 Oscillator Potential

For the oscillator potential

$$V_o = \frac{2R^2\omega^2 r^2}{(1-r^2)^2}$$

define the self-adjoint two-body Hamiltonian according to Theorem 2.12, where the space $M' \subset M = Q \times Q$ is defined by the inequality $\rho_2 < \frac{1}{2}\pi R$ (or equivalently by $r = \tan\frac{\rho}{2R} < 1$) and ρ_2 is the distance between particles. The domain for this operator is given by (2.32).

Equation (8.23) for this potential is a Fuchsian one with six singular points $0, \pm 1, \pm \mathbf{i}, \infty$ and corresponding characteristic exponents:

$$\begin{aligned}
\rho_\pm^{(0)} &= \frac{1}{2}\left(1 - q_1 - q_2 \pm \sqrt{(q_1+q_2-1)^2 + 32a}\right),\\
\rho_\pm^{(\infty)} &= \frac{1}{2}\left(1 - q_2 \pm \sqrt{(q_2-1)^2 + 32c}\right),\\
\rho_\pm^{(\mathbf{i})} &= \rho_\pm^{(-\mathbf{i})}\\
&= \frac{1}{4}q_1 + \frac{1}{2}q_2 \pm \frac{1}{2}\sqrt{\left(\frac{1}{2}q_1 + q_2\right)^2 + 8mER^2 + 4mR^4\omega^2 + 8(a-b+c)},\\
\rho_\pm^{(1)} &= \rho_\pm^{(-1)} = \frac{1}{2}\left(1 \pm \sqrt{1 + 4R^4 m\omega^2}\right).
\end{aligned}$$

Similarly to the previous section the function $f(r),\ r \in (0,1)$ should be $\sim r^{\rho_+^{(0)}}$ as $r \to +0$ and arguments for the one-body oscillator problem from Sect. 6.3.2 imply $f \sim r^{\rho_+^{(1)}}$ as $r \to 1-0$.

Fortunately, one can glue points $r = \pm 1$ together (as well as points $r = \pm\mathbf{i}$) by the change of the independent variable $r \to \zeta$, $\zeta = r^2$, which transforms the differential equation under consideration into the following Fuchsian differential equation with four singular points:

$$\begin{aligned}
f_{\zeta\zeta} &+ \frac{1 + q_1 + q_2 + (3-q_2)\zeta}{2\zeta(\zeta+1)} f_\zeta + \frac{2}{\zeta(\zeta+1)^2}\\
&\times\left(mR^2\left(E - \frac{2R^2\omega^2\zeta}{(\zeta-1)^2}\right) - \frac{a}{\zeta} - b - c\zeta\right) f = 0\,.
\end{aligned} \tag{8.35}$$

The singular points $-1, 0, 1, \infty$ of this equation form a harmonic quadruple and correspond, respectively, to the following characteristic exponents:

$$\rho_\pm^{(\mathbf{i})},\ \frac{1}{2}\rho_\pm^{(0)},\ \rho_\pm^{(1)},\ \frac{1}{2}\rho_\pm^{(\infty)}\,.$$

The same arguments as for the Coulomb problem leads to the conclusion that the only possibility to transform (8.35) to the hypergeometric equation via transformations (B.3), (B.4) and then using Theorem B.1 corresponds to the map of the unordered pair $(0,\infty)$ into the unordered pair $(0,2)$ by a Möbius transformation.

Without loss of generality, one can consider the substitution

$$t = \mu(\zeta) = \frac{2\zeta}{\zeta + 1}, \quad \mu : (-1, 0, 1, \infty) \to (\infty, 0, 1, 2) . \tag{8.36}$$

The interval under consideration for the variable t is again $(0, 1)$. Substitution (8.36) transforms (8.35) into (8.26) with

$$\mathcal{A}(t) = \frac{(q_1 + 2q_2 + 2)t - 2(q_1 + q_2 + 1)}{2t(t-2)},$$
$$\mathcal{B}(t) = \frac{2}{t(t-2)} \left(mR^2 \left(E + \frac{R^2\omega^2 t(t-2)}{2(t-1)^2} \right) - \frac{2a}{t} + a - b + \frac{ct}{t-2} \right) .$$

Define a function $w(t)$ by

$$w(t) = t^{-\frac{1}{2}\rho_+^{(0)}} (t-1)^{-\rho_+^{(1)}} (t-2)^{-\frac{1}{2}\rho_+^{(\infty)}} f(t) .$$

It satisfies Heun equation (B.14), where

$$\alpha = \frac{1}{2}\rho_+^{(0)} + \rho_+^{(1)} + \frac{1}{2}\rho_+^{(\infty)} + \rho_+^{(\mathrm{i})},\ \beta = \frac{1}{2}\rho_+^{(0)} + \rho_+^{(1)} + \frac{1}{2}\rho_+^{(\infty)} + \rho_-^{(\mathrm{i})},\ d = 2,$$
$$\gamma = 1 + \frac{1}{2}\left(\rho_+^{(0)} - \rho_-^{(0)}\right),\ \delta = 1 + \rho_+^{(1)} - \rho_-^{(1)},\ \varepsilon = 1 + \frac{1}{2}\left(\rho_+^{(\infty)} - \rho_-^{(\infty)}\right) .$$

Here the expression $t^{-\frac{1}{2}\rho_+^{(0)}} (t-1)^{-\rho_+^{(1)}} (t-2)^{-\frac{1}{2}\rho_+^{(\infty)}}$ denotes the function holomorphic on the domain $\mathbb{C} \setminus ((-\infty, 0] \cup [1, +\infty))$ and real for real $t \in (0, 1)$. Restrictions on asymptotics of the function $f(r)$ near the points $r = 0, 1$ are equivalent to the boundedness of the function $w(t)$ near the points $t = 0, 1$.

Calculation, similar to (8.27), yields the following value of accessory parameter q for (B.14)

$$q = -2mR^2E + 2b + (q_1 + q_2 + 1)\left(\rho_+^{(1)} + \frac{1}{4}\rho_+^{(\infty)}\right)$$
$$+ 2\rho_+^{(0)}\rho_+^{(1)} + \frac{1}{2}\rho_+^{(0)}\rho_+^{(\infty)} + \frac{1}{4}(q_2 + 1)\rho_+^{(0)} .$$

Condition (8.28) of Theorem B.1 is again equivalent to (8.30). Condition (8.29) of the same Theorem can be written as

$$\alpha\beta - q = \rho_+^{(1)}\left(\rho_+^{(\infty)} - \rho_+^{(0)} - \frac{1}{2}q_1\right) = 0 ,$$

which is again a consequence of (8.28).

Suppose till the end of this section that condition (8.28) is valid. Thus, we are in the situation of the first case of Theorem B.1 and, changing the independent variable t by a new one z according to (8.33), one gets hypergeometric equation (8.34) with

$$\widetilde{\alpha} = \frac{1}{2}\alpha = \frac{1}{4}\left(2 + \sqrt{(q_2-1)^2+32a} + \sqrt{1+4R^4m\omega^2} + s\right),$$
$$\widetilde{\beta} = \frac{1}{2}\beta = \frac{1}{4}\left(2 + \sqrt{(q_2-1)^2+32a} + \sqrt{1+4R^4m\omega^2} - s\right),$$
$$\widetilde{\gamma} = \gamma = 1 + \frac{1}{2}\sqrt{(q_2-1)^2+32a},$$

where $s = \sqrt{(\frac{1}{2}q_1+q_2)^2 + 8mER^2 + 4mR^4\omega^2 + 16a - 8b}$. The interval $(0,1)$ $\ni t$ corresponds to the interval $(0,1) \ni z$, therefore the requirement on asymptotic of the function $f(t)$ near the point $t=0$ implies

$$w(z) = F(\widetilde{\alpha}, \widetilde{\beta}; \widetilde{\gamma}; z)\ .$$

Also, due to

$$\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta} = -\frac{1}{2}\sqrt{1+4R^4m\omega^2} < 0,\ \operatorname{Re}\widetilde{\alpha} > 0$$

and (B.10), the requirement on asymptotic of the function $f(t)$ near the point $t=1$ implies

$$\widetilde{\beta} = -k,\ k = 0,1,2,\ldots$$

This leads to the energy levels

$$\begin{aligned} E_k = {} & \frac{1}{8mR^2}\left(\left(4k+2+\sqrt{(q_2-1)^2+32a}\right)^2 \right. \\ & \left. -\frac{1}{2}q_1q_2 - q_2^2 - \frac{1}{2}q_1 - 16a + 8b + 1\right) \\ & + \frac{\omega}{2\sqrt{m}}\left(4k+2+\sqrt{(q_2-1)^2+32a}\right)\sqrt{1+\frac{1}{4R^4m^2}},\ k = 0,1,2,\ldots \end{aligned}$$

Particularly, for the real spheres $\mathbf{S}^n$ and projective spaces $\mathbf{P}^n(\mathbb{R})$ in the case $q_1 = 0$, $q_2 = n-1$, $a = c$, which corresponds to cases 1,4 of Propositions 8.3, 8.5 and cases 1,2,5,8 of Proposition 8.4, one gets

$$\begin{aligned} E_k = {} & \frac{1}{8mR^2}\left(\left(4k+2+\sqrt{(n-2)^2+32a}\right)^2 - (n-1)^2 - 16a + 8b + 1\right) \\ & + \frac{\omega}{2\sqrt{m}}\left(4k+2+\sqrt{(n-2)^2+32a}\right)\sqrt{1+\frac{1}{4R^4m^2}},\ k = 0,1,2,\ldots \end{aligned}$$

The expression for the corresponding radial eigenfunctions (up to an arbitrary constant nonzero factor) is

$$f_k(r) = \frac{r^{\rho_+^{(0)}}(r^2-1)^{\rho_+^{(1)}}}{(r^2+1)^{\frac{1}{2}\rho_+^{(0)}+\rho_+^{(1)}+\frac{1}{2}\rho_+^{(\infty)}}}\sum_{j=0}^{k}\frac{(-1)^j}{j!(k-j)!}\frac{(\widetilde{\alpha})_j}{(\widetilde{\gamma})_j}\frac{4^j r^{2j}}{(r^2+1)^{2j}},$$

where $\rho_+^{(0)}$, $\rho_+^{(1)}$, $\rho_+^{(\infty)}$, $\widetilde{\alpha}$ and $\widetilde{\gamma}$ are given by above formulas for $E = E_k$.

8.4 The Problem of the Discrete Spectrum on Noncompact Spaces

The approach of this chapter to the calculation of some energy series for the two-body problem on compact two-point homogeneous spaces can not be applied for noncompact two-point homogeneous spaces, since there are no finite-dimensional unitary representations of noncompact Lie groups.

Conjecture 8.1. *There are no discrete spectrum for the two-body Hamiltonian with a central potentials in noncompact two-point homogeneous spaces.*

As an indirect evidence, we recall that in Euclidean space, the two-body Hamiltonian with a central potentials has no discrete spectrum unless the part corresponding to the center of mass motion is separated. This fact is not stressed in quantum mechanical textbooks. Indeed, the two-body Hamiltonian in Euclidean space $\mathbf{E}^n$ can be expressed in the following form [99]:

$$\widehat{H} = \widehat{H}_r + \widehat{H}_c, \quad \widehat{H}_r = -\frac{1}{2m} \triangle_r + U, \quad \widehat{H}_c = -\frac{1}{2m} \triangle_c,$$

where $\triangle_r$ and $\triangle_c$ are the Laplacians respectively corresponding to the relative motion of the particles and the center of mass motion. Decomposing the space of states into the tensor product $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_c$ and applying the spectral theorem [144] to the commutative operators: $\widehat{H}_r$ on the space $\mathcal{H}_r$ and $\widehat{H}_c$ on the space $\mathcal{H}_c$, one sees that these operators are unitary equivalent to the respective multiplications by a measurable real-valued function f_r on the space $\mathcal{L}^2(M_r, d\mu_r)$ and by a measurable real-valued function f_c on $\mathcal{L}^2(M_c, d\mu_c)$, where $\mu_r(M_r) < \infty$, $\mu_c(M_c) < \infty$. Therefore, the operator $\widehat{H}$ is unitary equivalent to the operator of multiplication by the function $f_r + f_c$ acting in the space $\mathcal{L}^2(M_r \times M_c,\, d\mu_r \times d\mu_c)$.

Eigenvalues of multiplication operators can be described as follows. Let M be a measurable set with a measure μ such that $\mu(M) < \infty$, f be a measurable complex-valued function on M and H_f be the multiplication operator:

$$H_f \psi = f\psi, \text{ where } \psi \in \mathrm{Dom}(H_f) := \left(\varphi \in \mathcal{L}^2(M, d\mu) \middle| \int_M |f\varphi|^2 d\mu < \infty \right) .$$

A complex number λ is an eigenvalue for H_f iff $\mu\left(f^{-1}(\lambda)\right) > 0$. The corresponding eigenfunction is the characteristic function of the set $f^{-1}(\lambda)$ up to a nonzero multiplicative constant.

Since the spectrum of the operator $\widehat{H}_c$ (corresponding to a free motion) is purely continuous, it holds $\mu_c\left(f_c^{-1}(\lambda)\right) = 0$ for any $\lambda \in \mathbb{C}$. One should prove that

$$(\mu_r \times \mu_c)(\Sigma_\lambda) = 0, \ \forall \lambda \in \mathbb{C},$$
$$\text{where } \Sigma_\lambda := ((x, y) \in M_r \times M_c \,|\, f_r(x) + f_c(y) = \lambda) \ .$$

Let χ_λ be the characteristic function of the set Σ_λ. Since the measures of M_r and M_c are finite, the Fubini theorem [144] applied to χ_λ implies

$$(\mu_r \times \mu_c)(\Sigma_\lambda) = \int\limits_{M_r \times M_c} \chi_\lambda \, d\mu_r(x) \times d\mu_c(y) = \int\limits_{M_r} \int\limits_{M_c} \chi_\lambda d\mu_c(y) \, d\mu_r(x)$$
$$= \int\limits_{M_r} \mu_c(f_c^{-1}(\lambda - f_r(x))) d\mu_r(x) = \int\limits_{M_r} 0 \cdot d\mu_r(x) = 0 \, .$$

Thus, the operator $\widehat{H}$ has no eigenvalues. It is therefore natural to expect that the two-body Hamiltonian on noncompact two-point homogeneous spaces with nonseparable variables also has no eigenvalues.

A

Calculations of Commutator Relations for Algebras of Invariant Differential Operator

In this appendix we shall illustrate main ideas of calculating commutator relations in Chap. 3. We shall obtain here some relations requiring minimal calculations.

It is not difficult to verify the following equalities for elements A, B, C of an arbitrary associative algebra:

$$[A, \{B, C\}] = \{[A, B], C\} + \{B, [A, C]\}, \tag{A.1}$$

$$\{\{A, B\}, C\} - \{A, \{B, C\}\} = [B, [A, C]], \tag{A.2}$$

$$\{\{A, B\}, C\} = 2\{B, C\}A + \{[A, B], C\} + \{[A, C], B\} + [B, [A, C]]\ . \tag{A.3}$$

Let start from commutator relations (3.15) from Sect. 3.2.2. Let operators $D_0, \dots, D_{10}$ are defined as in Sect. 3.2.1. First let us consider the commutator $[D_1, D_4]$. When $C = B$, from (A.1) one has $[A, B^2] = \{[A, B], B\}$. This implies:

$$[D_1, D_4] = \{[D_1, \Upsilon_{12}], \Upsilon_{12}\} + \{[D_1, \Omega_{12}], \Omega_{12}\} + \{[D_1, \Theta_{12}], \Theta_{12}\}\ .$$

Using (1.12) and (A.1) again, one gets

$$[D_1, \Upsilon_{12}] = \Box_1,\ [D_1, \Omega_{12}] = \Box_2,\ [D_1, \Theta_{12}] = \Box_3\ .$$

Thus

$$[D_1, D_4] = \{\Box_1, \Upsilon_{12}\} + \{\Box_2, \Omega_{12}\} + \{\Box_3, \Theta_{12}\} = 2D_7\ . \tag{A.4}$$

Using the permutation of coordinates z_1 and z_2 (or equivalently the automorphism $\sigma \circ \zeta_\pi$, see Sect. 3.2.1), we obtain from (A.4):

$$[D_2, D_4] = -2D_7\ .$$

Suppose now that we already know the expressions for commutators

$$[D_0, D_1],\ [D_0, D_3],\ [D_0, D_7],\ [D_1, D_2],\ [D_1, D_4],\ [D_1, D_5],\ [D_1, D_6],\ [D_1, D_7],$$
$$[D_1, D_8],\ [D_2, D_6],\ [D_3, D_4],\ [D_3, D_6],\ [D_4, D_5],\ [D_4, D_6],\ [D_4, D_8]$$

from (3.15).

Then from the Jacobi identity and (A.1) one has

$$\begin{aligned}
0 &= [D_1,[D_8,D_4]] + [D_4,[D_1,D_8]] + [D_8,[D_4,D_1]] \\
&= [D_1, \frac{1}{2}\{D_2 - D_1, D_6\} + \{D_0, D_7\}] \\
&+ [D_4, n(n-1)D_6 - \frac{1}{2}\{D_3, D_5\} + \frac{3}{4}D_3 - \frac{1}{2}\{D_1, D_6\}] - 2[D_8, D_7] \\
&= \frac{1}{2}\{[D_1,D_2],D_6\} - \frac{1}{2}\{D_1 - D_2, [D_1, D_6]\} \\
&+ \{[D_1,D_0],D_7\} + \{D_0,[D_1,D_7]\} + n(n-1)[D_4,D_6] \\
&- \frac{1}{2}\{[D_4,D_3],D_5\} - \frac{1}{2}\{D_3,[D_4,D_5]\} + \frac{3}{4}[D_4,D_3] \\
&- \frac{1}{2}\{D_1,[D_4,D_6]\} - \frac{1}{2}\{[D_4,D_1],D_6\} - 2[D_8,D_7] = -\frac{1}{2}\{\{D_3,D_0\},D_6\} \\
&- \{D_4,D_6\} + \frac{1}{2}\{D_2 - D_1, D_8\} + \{D_3,D_7\} + \{D_0, n(n-1)D_4 \\
&- \frac{1}{2}\{D_3,D_6\} - \frac{1}{2}\{D_1,D_4\} + \frac{3}{8}(D_1 - D_2) + D_9 + D_{10}\} \\
&- n(n-1)\{D_0,D_4\} + \frac{3}{2}n(n-1)D_0 + \{D_3,\{D_6,D_0\}\} \\
&+ \frac{1}{2}\{D_1,\{D_0,D_4\}\} - \frac{3}{4}\{D_1,D_0\} + \{D_7,D_6\} \\
&- 2[D_8,D_7] = \frac{1}{2}\{D_2 - D_1, D_8\} + \{D_3,D_7\} + \frac{3}{8}\{D_0, D_1 - D_2\} \\
&+ \{D_0, D_9 + D_{10}\} + \frac{3}{2}n(n-1)D_0 - \frac{3}{4}\{D_0,D_1\} - 2[D_8,D_7]\ .
\end{aligned}$$

In the last equality we took into account the formulas

$$\begin{aligned}
&\{\{D_6,D_0\},D_3\} - \{D_6,\{D_0,D_3\}\} + \{\{D_0,D_6\},D_3\} \\
&- \{D_0,\{D_6,D_3\}\} = [D_0,[D_6,D_3]] + [D_6,[D_0,D_3]] \\
&= -[D_0,D_7] - \frac{1}{2}[D_6, D_2 - D_1] = D_8 + \frac{1}{2}(-D_8 - D_8) = 0, \\
&\{\{D_0,D_4\},D_1\} - \{D_0,\{D_4,D_1\}\} = [D_4,[D_0,D_1]] = [D_4,D_3] = 0,
\end{aligned}$$

which are consequences of (A.2).

Thus, one gets:

$$\begin{aligned}
[D_7,D_8] &= \frac{1}{4}\{D_1 - D_2, D_8\} - \frac{1}{2}\{D_3,D_7\} \\
&+ \frac{3}{16}\{D_0, D_1 + D_2\} - \frac{1}{2}\{D_0, D_9 + D_{10}\} \\
&- \frac{3}{4}n(n-1)D_0\ .
\end{aligned}$$

Now let us demonstrate the calculation modulo $(U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$. Let $D_0, \dots, D_3$ be generators of $\mathrm{Diff}(\mathbf{S}^n_{\mathbf{S}})$, $n \geqslant 3$ and $\mathfrak{g} = \mathfrak{so}(n+1)$, $\mathfrak{k}_0 = \mathfrak{so}(n-1)$, $K_0 = \mathrm{SO}(n-1)$. Then from (A.1) we obtain:

$$
\begin{aligned}
[D_1, D_3] &= -8 \sum_{k,l=3}^{n+1} \left(\{\{\Psi_{1k}, [\Psi_{1k}, \Psi_{1l}]\}, \Psi_{2l}\} + \{\Psi_{1l}, \{\Psi_{1k}, [\Psi_{1k}, \Psi_{2l}]\}\}\right) \\
&= 4 \sum_{k,l=3}^{n+1} \left(\{\{\Psi_{1k}, \Psi_{kl}\}, \Psi_{2l}\} + \delta_{kl}\{\Psi_{1l}, \{\Psi_{1k}, \Psi_{12}\}\}\right) \\
&= 4 \sum_{\substack{k,l=3 \\ k\neq l}}^{n+1} \{\{\Psi_{kl}, \Psi_{1k}\}, \Psi_{2l}\} + 4 \sum_{k=3}^{n+1} \{\Psi_{1k}, \{\Psi_{1k}, \Psi_{12}\}\} \, . \qquad \text{(A.5)}
\end{aligned}
$$

From formula (A.3) and commutator relations (1.12) one gets:

$$
\begin{aligned}
&\sum_{\substack{k,l=3 \\ k\neq l}}^{n+1} \{\{\Psi_{kl}, \Psi_{1k}\}, \Psi_{2l}\} = \sum_{\substack{k,l=3 \\ k\neq l}}^{n+1} (2\{\Psi_{1k}, \Psi_{2l}\}\Psi_{kl} \\
&+ \{[\Psi_{kl}, \Psi_{1k}], \Psi_{2l}\} + \{[\Psi_{kl}, \Psi_{2l}], \Psi_{1k}\} + [\Psi_{1k}, [\Psi_{kl}, \Psi_{2l}]]) \\
&\equiv \sum_{\substack{k,l=3 \\ k\neq l}}^{n+1} \left(-\frac{1}{2}\{\Psi_{1l}, \Psi_{2l}\} + \frac{1}{2}\{\Psi_{2k}, \Psi_{1k}\} + \frac{1}{2}[\Psi_{1k}, \Psi_{2k}]\right) \quad \mathrm{mod} \ \left(U(\mathfrak{g})\mathfrak{k}_0\right)^{K_0} \\
&= -\frac{1}{4} \sum_{\substack{k,l=3 \\ k\neq l}}^{n+1} \Psi_{12} = -\frac{(n-1)(n-2)}{4}\Psi_{12} = \frac{(n-1)(n-2)}{8} D_0 \, .
\end{aligned}
$$

Formula (A.2) gives:

$$
\begin{aligned}
&\sum_{k=3}^{n+1} \{\Psi_{1k}, \{\Psi_{1k}, \Psi_{12}\}\} = \sum_{k=3}^{n+1} (\{\{\Psi_{1k}, \Psi_{1k}\}, \Psi_{12}\} \\
&- [\Psi_{1k}, [\Psi_{1k}, \Psi_{12}]]) = 2 \left\{ \sum_{k=3}^{n+1} \Psi_{1k}^2, \Psi_{12} \right\} - \frac{1}{2} \sum_{k=3}^{n+1} [\Psi_{1k}, \Psi_{2k}] \\
&= -\frac{1}{4}\{D_0, D_1\} + \frac{1}{4} \sum_{k=3}^{n+1} \Psi_{12} = -\frac{1}{4}\{D_0, D_1\} - \frac{n-1}{8} D_0 \, .
\end{aligned}
$$

Finally, from (A.5) we obtain:

$$
\begin{aligned}
[D_1, D_3] &= -\{D_0, D_1\} - \frac{n-1}{2} D_0 + \frac{(n-1)(n-2)}{2} D_0 \\
&= -\{D_0, D_1\} + \frac{(n-1)(n-3)}{2} D_0 \, .
\end{aligned}
$$

Calculations of the commutator $[D_1, D_3]$ for algebras $\mathrm{Diff}(\mathbf{P}^n(\mathbb{H})_{\mathbf{S}})$ and $\mathrm{Diff}(\mathbf{P}^n(\mathbb{C})_{\mathbf{S}})$ are analogous, but much longer.

Let us demonstrate calculations in octonionic case by one example. Below indices i, j vary from 0 to 7. Suppose that $D_0, \ldots, D_9$ are generators of $\mathrm{Diff}\left(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}\right)$ and it holds $\mathfrak{g} = \mathfrak{f}_4$, $\mathfrak{k}_0 = \mathfrak{spin}(7)$, $K_0 = \mathrm{Spin}(7)$ (see Sect. 3.5). Then from (A.1) and Proposition 3.4 one gets:

$$[D_1, D_3] = \frac{1}{2}\sum_{i,j}(\{\{[e_{\lambda,i}, e_{\lambda,j}], e_{\lambda,i}\}, f_{\lambda,j}\} + \{e_{\lambda,j}, \{e_{\lambda,i}, [e_{\lambda,i},, f_{\lambda,j}]\}\})$$
$$= \frac{1}{8}\sum_{i\neq j}\big\{\{\varkappa C_{2,\bar{e}_i,\bar{e}_j}, e_{\lambda,i}\}, f_{\lambda,j}\big\} - \frac{1}{4}\sum_i \{e_{\lambda,i}, \{e_{\lambda,i}, \Lambda\}\}$$
$$- \frac{1}{4}\sum_{i\neq j}\big\{e_{\lambda,j}, \{e_{\lambda,i}, e_{2\lambda,e_i\bar{e}_j}\}\big\} \ .$$

Formulas (A.3), (1.44), (1.32) and Proposition 3.4 imply:

$$\frac{1}{8}\sum_{i\neq j}\big\{\{\varkappa C_{2,\bar{e}_i,\bar{e}_j}, e_{\lambda,i}\}, f_{\lambda,j}\big\}$$
$$= \frac{1}{8}\sum_{i,j}\big(2\{e_{\lambda,i}, f_{\lambda,j}\}\varkappa C_{2,\bar{e}_i,\bar{e}_j} + \big\{[\varkappa C_{2,\bar{e}_i,\bar{e}_j}, e_{\lambda,i}], f_{\lambda,j}\big\}$$
$$+ \big(\big\{[\varkappa C_{2,\bar{e}_i,\bar{e}_j}, f_{\lambda,j}], e_{\lambda,i}\big\} + \big[e_{\lambda,i}, \big[\varkappa C_{2,\bar{e}_i,\bar{e}_j}, f_{\lambda,j}\big]\big]\big)$$
$$\equiv \sum_{i\neq j}\left(-\frac{1}{16}\left\{\operatorname{ad} Y_2\left(\varkappa C_{2,\bar{e}_i,\bar{e}_j}\big|_{\mathbb{C}a_2}\,\bar{e}_i\right), f_{\lambda,j}\right\}\right.$$
$$+\frac{1}{16}\left\{\operatorname{ad} Y_1\left(\varkappa C_{2,\bar{e}_i,\bar{e}_j}\big|_{\mathbb{C}a_1}\,e_j\right), e_{\lambda,i}\right\} + \frac{1}{16}\left[e_{\lambda,i}, \operatorname{ad} Y_1\left(\varkappa C_{2,\bar{e}_i,\bar{e}_j}\big|_{\mathbb{C}a_1}\,e_j\right)\right]$$
$$\left.+\frac{1}{2}\{e_{\lambda,i}, f_{\lambda,j}\}f_{2\lambda,e_i\bar{e}_j}\right) \mod (U(\mathfrak{g})\mathfrak{k}_0)^{K_0}$$
$$= \sum_{i\neq j}\left(-\frac{1}{4}\{\operatorname{ad} Y_2(\bar{e}_j), f_{\lambda,j}\} - \frac{1}{8}\{\operatorname{ad} Y_1(e_i), e_{\lambda,i}\} - \frac{1}{8}[e_{\lambda,i}, \operatorname{ad} Y_1(e_i)]\right.$$
$$\left.+\frac{1}{4}\left(\{f_{2\lambda,e_i\bar{e}_j}, \{e_{\lambda,i}, f_{\lambda,j}\}\} - [f_{2\lambda,e_i\bar{e}_j}, \{e_{\lambda,i}, f_{\lambda,j}\}]\right)\right)$$
$$= \sum_{i\neq j}\left(\frac{1}{2}\{e_{\lambda,j}, f_{\lambda,j}\} - \frac{1}{4}\{f_{\lambda,i}, e_{\lambda,i}\} - \frac{1}{4}[e_{\lambda,i}, f_{\lambda,i}]\right) + D_8$$
$$- \frac{1}{4}\sum_{i\neq j}\big(\big\{[f_{2\lambda,e_i\bar{e}_j}, e_{\lambda,i}], f_{\lambda,j}\big\} + \{e_{\lambda,i}, [f_{2\lambda,e_i\bar{e}_j}, f_{\lambda,j}]\}\big)$$
$$= D_8 + \sum_{i\neq j}\left(\frac{1}{4}\{e_{\lambda,j}, f_{\lambda,j}\} + \frac{1}{8}\Lambda + \frac{1}{8}\{e_{\lambda,e_i\bar{e}_j\cdot e_i}, f_{\lambda,j}\} - \frac{1}{8}\{e_{\lambda,i}, f_{\lambda,e_i\bar{e}_j\cdot e_j}\}\right)$$
$$= D_8 + \sum_{i\neq j}\left(\frac{1}{4}\{e_{\lambda,j}, f_{\lambda,j}\} - \frac{1}{8}\{e_{\lambda,e_j\bar{e}_ie_i}, f_{\lambda,j}\} - \frac{1}{8}\{e_{\lambda,i}, f_{\lambda,i}\}\right)$$
$$+ 7\Lambda = D_8 + 7D_0 \ .$$

Similarly, from (A.2) and Proposition 3.4 one gets:

$$- \frac{1}{4}\sum_i\{\{\Lambda, e_{\lambda,i}\}, e_{\lambda,i}\} = -\frac{1}{4}\sum_i(\{\Lambda, \{e_{\lambda,i}, e_{\lambda,i}\}\} + [e_{\lambda,i}, [\Lambda, e_{\lambda,i}]])$$
$$= -\frac{1}{2}\left\{\Lambda, \sum_i e_{\lambda,i}^2\right\} + \frac{1}{8}\sum_i[e_{\lambda,i}, f_{\lambda,i}] = -\frac{1}{2}\{D_0, D_1\} - \frac{1}{2}D_0 \ .$$

Also, it holds

$$\begin{aligned}
&-\frac{1}{4}\sum_{i\neq j}\{\{e_{2\lambda,e_i\bar{e}_j},e_{\lambda,i}\},e_{\lambda,j}\}\\
&=-\frac{1}{4}\sum_{i\neq j}\big(\{e_{2\lambda,e_i\bar{e}_j},\{e_{\lambda,i},e_{\lambda,j}\}\}+[e_{\lambda,i},[e_{2\lambda,e_i\bar{e}_j},e_{\lambda,j}]]\big)\\
&=-\frac{1}{8}\sum_{i\neq j}[e_{\lambda,i},f_{\lambda,e_i\bar{e}_j\cdot e_j}]=-\frac{1}{8}\sum_{i\neq j}[e_{\lambda,i},f_{\lambda,i}]=\frac{7\cdot 8}{2\cdot 8}\Lambda=\frac{7}{2}D_0,
\end{aligned}$$

since $e_{2\lambda,e_i\bar{e}_j}$ is antisymmetric and $\{e_{\lambda,i},e_{\lambda,j}\}$ is symmetric w.r.t. i,j.

Thus, one concludes that

$$[D_1,D_3]=D_8+(7-\frac{1}{2}+\frac{7}{2})D_0-\frac{1}{2}\{D_0,D_1\}=D_8-\frac{1}{2}\{D_0,D_1\}+10D_0\ .$$

B

Some Fuchsian Differential Equations

For convenience of references we collected here basic facts concerning some Fuchsian differential equations: the Riemannian equation and the reducibility of the Heun equation to the hypergeometric one.

The linear differential equation

$$w^{(n)}(z) + p_1(z)w^{(n-1)}(z) + \ldots + p_n(z)w(z) = 0 \tag{B.1}$$

on the Riemannian sphere $\overline{\mathbb{C}} = \mathbf{P}^1(\mathbb{C})$ with meromorphic coefficients $p_i(z)$, $i = 1, \ldots, n$ is *Fuchsian* [46] if for any $z_0 \in \overline{\mathbb{C}}$ its solutions has no more than a power growth as z tends to z_0 in some cone[1], not containing a whole neighborhood of its vertex z_0. A point z_0 is *regular* for this differential equation if all solutions of (B.1) are holomorphic in some neighborhood of z_0; otherwise z_0 is a *singular point*.

It is known [39, 50] that (B.1) is Fuchsian equation iff

$$p_i(z) = \frac{q_i(z)}{\prod\limits_{k=1}^{m} (z - z_k)^i}$$

for some finite potentially singular points $z_1, \ldots, z_m \in \mathbb{C}$ and polynomials $q_i(z)$ of degrees $\leqslant i(m-1)$. One can find *characteristic exponents* $\rho^{(z_k)}$ of (B.1) at the point z_k by substituting the expression $w(z) = (z - z_k)^{\rho^{(z_k)}}$ into (B.1) and keeping only leading terms as $z \to z_k$. This procedure gives an algebraic equation of the nth degree for $\rho^{(z_k)}$. Denote by $\rho_i^{(z_k)}$, $i = 1, \ldots, n$ its solutions for all points z_k, $k = 1, \ldots, m$. The substitution $w(z) = z^{-\rho^{(\infty)}}$ similarly gives characteristic exponents $\rho_1^{(\infty)}, \ldots, \rho_n^{(\infty)}$ at infinity. These characteristic exponents satisfy the *Fuchs identity*:

$$\sum_{i=1}^{n} \sum_{k=1}^{m+1} \rho_i^{(z_k)} = \frac{1}{2}(m-1)n(n-1) \ ,$$

where $\rho_i^{(z_{m+1})} := \rho_i^{(\infty)}$.

[1] In a neighborhood of the infinite point one should use the local coordinate $\zeta = 1/z$ instead of $z - z_0$.

One can find characteristic exponents also for a regular point. If a point $\tilde{z}$ is regular, then characteristic exponents for this point are $0, 1, \ldots, n-1$. The sufficient condition for the regularity of $\tilde{z}$ is the regularity of coefficients $p_i(z)$, $i = 1, \ldots, n$ at this point.

An information on singular points and corresponding characteristic exponents of (B.1) can be encoded in the *Riemann P-symbol* $P\{\mathcal{A}; z\}$, where the first row of a matrix $\mathcal{A}$ consists of singular points and other rows of $\mathcal{A}$ consist of corresponding characteristic exponents.

Equation (B.1) of the second order with three singular points is called the *Riemannian equation*. Coefficients of the Riemann equation are completely defined by its characteristic exponents. Equivalently, the Riemann equation is completely defined by its P-symbol. In this case the Fuchs identity looks like

$$\sum_{i=1}^{2}\sum_{k=1}^{3}\rho_i^{(z_k)} = 1$$

and there are only five independent characteristic values.

If all three singular points z_1, z_2, z_3 are finite, then the Riemannian equation has the form [50]:

$$\begin{aligned} w'' &+ \left(\frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_3}{z-z_3}\right) w' \\ &+ \left(\frac{B_1}{z-z_1} + \frac{B_2}{z-z_2} + \frac{B_3}{z-z_3}\right) \frac{w}{(z-z_1)(z-z_2)(z-z_3)} = 0, \end{aligned} \tag{B.2}$$

where $A_k = 1 - \rho_1^{(z_k)} - \rho_2^{(z_k)}$, $B_k = \rho_1^{(z_k)}\rho_2^{(z_k)}(z_k - z_{k-1})(z_k - z_{k+1})$. In the last equality indices are modulo 3.

There are two types of variable change, transforming any Fuchsian equation into another Fuchsian equation. The first one is a *linear-fractional (Möbius) transformation* of the independent variable:

$$z \to t,\ z = \frac{\alpha t + \beta}{\gamma t + \delta},\ \alpha\delta - \beta\gamma \neq 0\ . \tag{B.3}$$

By such transformation one can move three singular points into three arbitrary points of $\overline{\mathbb{C}}$ with the same characteristic exponents.

The second one is a linear transformation of the dependent variable

$$w(z) \to w_1(z) = \left(\frac{z-z_1}{z-z_2}\right)^q w(z), \tag{B.4}$$

which conserves singular points, but changes the characteristic exponents

$$\rho_i^{(z_1)} \to \rho_i^{(z_1)} + q,\ \rho_i^{(z_2)} \to \rho_i^{(z_2)} - q,\ i = 1, 2\ .$$

Using these transformation for the Riemannian equation one can move three singular points into the triple $(0, 1, \infty)$ such that $\rho_1^{(0)} = \rho_1^{(1)} = 0$. If one denote $\rho_1^{(\infty)} = \alpha, \rho_2^{(\infty)} = \beta$ and $\rho_2^{(0)} = 1 - \gamma$, then the Fuchs identity for this equation gives $\rho_2^{(1)} = \gamma - \alpha - \beta$ that corresponds to the *hypergeometric* or *Gauss equation*:

$$z(1-z)w''(z) + (\gamma - (\alpha+\beta+1)z)w'(z) - \alpha\beta w(z) = 0 \; . \tag{B.5}$$

The P-symbol of (B.5) is

$$P \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & ; z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix} \right\} .$$

Many quantum mechanical problems for constant curvature spaces can be reduced to this equation, while their Euclidean counterparts lead to its limiting cases, obtained from (B.5) by confluence of singular points (such equations are not Fuchsian).

We shall consider only solutions of (B.5) in the case $\gamma \neq -m$, $m \in \mathbb{N}$. Solutions of (B.1), corresponding to different characteristic exponents near some singular point are called *canonical solutions* near that point. The series

$$F(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \; |z| < 1 \tag{B.6}$$

where $(a)_n := a(a+1)\dots(a+n-1)$, $(a)_0 := 1$, is the canonical solution of (B.5), corresponding to the characteristic exponent $\rho_1^{(0)} = 0$. The function $F(\alpha, \beta; \gamma; z)$, defined by (B.6) for $|z| < 1$, can be analytically continued for $z \in \mathbb{C}\backslash(1, +\infty)$ by different ways, for example using formulas (B.7)-(B.12) below or integral representations [1, 50].

Evidently, it holds $F(\alpha, \beta; \gamma; z) = F(\beta, \alpha; \gamma; z)$. If $\alpha = -m$ or $\beta = -m$, $m = 0, 1, 2, \dots$, then $F(\alpha, \beta; \gamma; z)$ is a polynomial of degree m. Also, the function $F(\alpha, \beta; \gamma; z)$ has a pole w.r.t γ at $\gamma = -m$, $m \in \mathbb{N}$ and

$$\lim_{\gamma \to -m} \frac{F(\alpha, \beta; \gamma; z)}{\Gamma(\gamma)} = \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(m+1)!} z^{m+1} F(\alpha+m+1, \beta+m+1; m+2; z)$$

(see [1]), where Γ is the *gamma-function*, defined as the analytic continuation for $z \in \mathbb{C}$ of the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \; \mathrm{Re}\, z > 0 \; .$$

The function Γ has no zeros and has poles of the first order at the points $z = -m$, $m = 0, 1, 2, \dots$. Its logarithmic derivative $\psi_\Gamma(z) := \Gamma'(z)/\Gamma(z)$ also has poles of the first order at the same points.

Another canonical solution of (B.5), corresponding to the characteristic exponent $\rho_2^{(0)} = 1 - \gamma$ for $\gamma \notin \mathbb{N}$, is

$$z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z) \; .$$

Canonical solutions near the singular point $z = 1$ are

$$F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)$$

and if $\gamma-\alpha-\beta\notin\mathbb{Z}$ also

$$(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z)\ .$$

Near the singular point $z=\infty$ canonical solutions are

$$z^{-\alpha}F\left(\alpha,\alpha-\gamma+1;\alpha-\beta+1;\frac{1}{z}\right),\ z^{-\beta}F\left(\beta,\beta-\gamma+1;\beta-\alpha+1;\frac{1}{z}\right)$$

if $\alpha-\beta\notin\mathbb{Z}$. If $\alpha-\beta\in\mathbb{Z}$, then only one of these expressions is a canonical solution: the first if $\alpha-\beta>0$ or the second if $\alpha-\beta<0$.

There are expansions of $F(\alpha,\beta;\gamma;z)$ through canonical solutions near the singular points $z=1$ and $z=\infty$ [1, 68], important for spectral problems. The first one is

$$\begin{aligned}F(\alpha,\beta;\gamma;z)&=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)\\&+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z),\\&|\arg(1-z)|<\pi\end{aligned} \tag{B.7}$$

if $\gamma-\alpha-\beta\notin\mathbb{Z}$. For $\gamma-\alpha-\beta\in\mathbb{Z}$ every summand at the right hand side of (B.7) is singular and it holds for $m=0,1,2,\ldots$

$$\begin{aligned}F(\alpha,\beta;\alpha+\beta+m;z)&=\frac{\Gamma(m)\Gamma(\alpha+\beta+m)}{\Gamma(\alpha+m)\Gamma(\beta+m)}\sum_{n=0}^{m-1}\frac{(\alpha)_n(\beta)_n}{n!(1-m)_n}(1-z)^n\\&-\frac{\Gamma(\alpha+\beta+m)}{\Gamma(\alpha)\Gamma(\beta)}(z-1)^m\sum_{n=0}^{\infty}\frac{(\alpha+m)_n(\beta+m)_n}{n!(n+m)!}(1-z)^n\\&\times(\ln(1-z)-\psi_\Gamma(n+1)-\psi_\Gamma(n+m+1)\\&+\psi_\Gamma(\alpha+n+m)+\psi_\Gamma(\beta+n+m)),\end{aligned} \tag{B.8}$$

$$\begin{aligned}&F(\alpha,\beta;\alpha+\beta-m;z)\\&=\frac{\Gamma(m)\Gamma(\alpha+\beta-m)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{-m}\sum_{n=0}^{m-1}\frac{(\alpha-m)_n(\beta-m)_n}{n!(1-m)_n}(1-z)^n\\&-\frac{(-1)^m\Gamma(\alpha+\beta-m)}{\Gamma(\alpha-m)\Gamma(\beta-m)}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!(n+m)!}(1-z)^n\\&\times(\ln(1-z)-\psi_\Gamma(n+1)-\psi_\Gamma(n+m+1)+\psi_\Gamma(\alpha+n)+\psi_\Gamma(\beta+n)),\\&|\arg(1-z)|<\pi,\ |1-z|<1\ .\end{aligned} \tag{B.9}$$

In the case $\mathrm{Re}(\gamma-\alpha-\beta)<0$, formulas (B.7) – (B.9) imply

$$\lim_{z\to1}F(\alpha,\beta;\gamma;z)(1-z)^{\alpha+\beta-\gamma}=\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\ . \tag{B.10}$$

The expansion of $F(\alpha,\beta,\gamma,z)$ through canonical solutions near the point $z = \infty$ is

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-z)^{-\alpha}F\left(\alpha,\alpha-\gamma+1;\alpha-\beta+1;\frac{1}{z}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)}(-z)^{-\beta}F\left(\beta,\beta-\gamma+1;\beta-\alpha+1;\frac{1}{z}\right),\ |\arg(-z)|<\pi, \tag{B.11}$$

if $\alpha-\beta\notin\mathbb{Z}$. If $\alpha-\beta\in\mathbb{Z}$, then every summand at the right hand side of (B.11) is singular. In this case:

$$F(\alpha,\alpha+m;\gamma;z) = F(\alpha+m,\alpha;\gamma;z) = \frac{\Gamma(\gamma)(-z)^{-\alpha}}{\Gamma(\alpha+m)}\sum_{n=0}^{m-1}\frac{\Gamma(m-n)(\alpha)_n}{n!\Gamma(\gamma-\alpha-n)}z^{-n}$$
$$+\frac{\Gamma(\gamma)(-z)^{-\alpha-m}}{\Gamma(\alpha+m)\Gamma(\gamma-m)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n+m}(1-\gamma+\alpha)_{n+m}}{n!(n+m)!}z^{-n} \tag{B.12}$$
$$\times\left(\ln(-z)+\psi_\Gamma(n+m+1)+\psi_\Gamma(n+1)-\psi_\Gamma(\alpha+n+m)\right.$$
$$\left.-\psi_\Gamma(\gamma-\alpha-n-m)\right),$$
$$|\arg(-z)|<\pi,\ |z|>1,\ \gamma-\alpha\notin\mathbb{Z}\,.$$

The corresponding formula for $\gamma-\alpha\in\mathbb{Z}$ can be obtained by taking the limit as $\gamma-\alpha\to k\in\mathbb{Z}$ in (B.12).

From (B.11) – (B.12) one sees that for $\operatorname{Re}\alpha > \operatorname{Re}\beta$ it holds

$$\lim_{z\to-\infty} F(\alpha,\beta;\gamma;z)(-z)^{\beta} = \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}\,. \tag{B.13}$$

The Fuchsian equation (B.1) of the second order with four singular points by transformations (B.3) and (B.4) can be reduced to the *Heun equation*

$$w''(t)+\left(\frac{\gamma}{t}+\frac{\delta}{t-1}+\frac{\varepsilon}{t-d}\right)w'(t)+\frac{\alpha\beta t-q}{t(t-1)(t-d)}w(t)=0, \tag{B.14}$$

where $0,1,d,\infty$ are its four singular points ($d\neq 0,1,\infty$) and $\alpha+\beta-\gamma-\delta-\varepsilon+1=0$. The corresponding P-symbol is

$$P\begin{Bmatrix} 0 & 1 & d & \infty & \\ 0 & 0 & 0 & \alpha & ;t \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta & \end{Bmatrix}.$$

Note that the *accessory parameter* q does not arise in this P-symbol.

The theory of the Heun equation is much less explicit than the theory of the Riemannian equation. In particular, there are no explicit expressions of canonical solutions near different singular points through each other. Therefore, there are only approximate methods for solving spectral problems connected with the Heun equation, using continued fractions (see for example [176] and references therein).

The substitution $z = P(t)$ for a rational function P transforms (B.1) into another Fuchsian equation with generally a greater number of singular points. Therefore, sometimes the inverse transformation can decrease the number of singular points of a Fuchsian equation.[2]

At the present time there is no a general theory of such reduction. However, in [113] there were classified all Heun equations (B.14) that can be obtained by a substitution $z = P(t)$ from the hypergeometric equation (B.5). By the inverse transformation these Heun equations are reduced to the hypergeometric equations.

The first condition for existing such reduction is the position of the point d. Let

$$(z_1, z_2, z_3, z_4)_{c.r.} := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

be the *cross-ratio* of four pairwise distinct points from $\overline{\mathbb{C}}$. It is well known that a cross-ration is invariant under Möbius transformations. The group $\mathfrak{S}_4$, permuting points z_1, z_2, z_3 and z_4, acts on their cross-ration. The cross-ration orbit $\mathcal{O}_{\mathfrak{S}_4}(s)$ of $s := (z_1, z_2, z_3, z_4)_{c.r.}$ consists of points $s, 1-s, 1/s, 1/(1-s), s/(s-1), (s-1)/s \in \overline{\mathbb{C}}$.

In general position this orbit consists of six points, but there are two exceptional cases: the orbit $-1, \frac{1}{2}, 2$ and the orbit $\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$. If $(z_1, z_2, z_3, z_4)_{c.r.} \in (-1, \frac{1}{2}, 2)$, then (z_1, z_2, z_3, z_4) is a *harmonic quadruple*. If $(z_1, z_2, z_3, z_4)_{c.r.} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$, then (z_1, z_2, z_3, z_4) is an *equianharmonic quadruple*.

Points of a harmonic quadruple lie on a circle or on a line. By a Möbius transformation they can be mapped into vertices of a square in $\mathbb{C}$. If (z_1, z_2, z_3, ∞) is a harmonic quadruple, then (z_1, z_2, z_3) are collinear, equally spaced points. If (z_1, z_2, z_3, ∞) is an equianharmonic quadruple, then (z_1, z_2, z_3) are vertices of an equilateral triangle in $\mathbb{C}$.

Theorem B.1 ([113]). *All cases, when nontrivial Heun equation* (B.14) *(i.e., $\alpha\beta \neq 0$ or $q \neq 0$) can be obtained from the hypergeometric one* (B.5) *by the rational substitution $z = P(t)$, are as follows.*

1. *Harmonic case: $d \in \mathcal{O}_{\mathfrak{S}_4}(2)$. Suppose $d = 2$,*[3] *then $q/(\alpha\beta)$ must be equal* 1*, and characteristic exponents of the points $t = 0$ and $t = d = 2$ must be the same, i.e., $\gamma = \varepsilon$. The function $P(t)$ is a degree-*2 *polynomial and can be chosen as $P(t) = t(2-t) = 1 - (t-1)^2$. It maps $t = 0, 2$ to $z = 0$ and $t = 1$ to $z = 1$.*[4]
   *If additionally $1 - \delta = 2(1 - \gamma)$, then $P(t)$ can be chosen also as degree-*4 *polynomial $4(t(2-t) - \frac{1}{2})^2$, which maps $t = 0, 1, 2$ to $z = 1$.*
2. *$d \in \mathcal{O}_{\mathfrak{S}_4}(4)$. Suppose $d = 4$, then $q/(\alpha\beta)$ must be equal* 1*, characteristic exponents of the point $t = 1$ must be double those of the point $t = d = 4$,*

[2] Generally, the inverse transformation does not conserve the Fuchs class of differential equations.

[3] If $d \in \mathcal{O}_{\mathfrak{S}_4}(s)$, then the quadruple $(0, 1, d, \infty)$ can be mapped into the quadruple $(0, 1, s, \infty)$ by a Möbius transformation, which transforms also parameters of (B.14).

[4] This transformation was found already in [96].

i.e., $1-\delta = 2(1-\varepsilon)$, *and* $t = 0$ *must have characteristic exponents* $0, 1/2$, *i.e.,* $\gamma = \frac{1}{2}$. *The function* $P(t)$ *is a degree-*3 *polynomial and can be chosen as* $(t-1)^2(1-\frac{t}{4})$. *It maps* $t = 0$ *to* $z = 1$ *and* $t = 1, 4$ *to* $z = 0$.

3. *Equianharmonic case:* $d \in \mathcal{O}_{\mathfrak{S}_4}(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i})$. *Characteristic exponents of the points* $t = 0, 1, d$ *are the same, i.e.,* $\gamma = \delta = \varepsilon$. *Suppose* $d = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$, *then* $q/(\alpha\beta)$ *must be equal* $\frac{1}{2} + \frac{\sqrt{3}}{6}\mathbf{i}$. *The function* $P(t)$ *is a degree-*3 *polynomial and can be chosen as* $\left(1 - t/(\frac{1}{2} + \frac{\sqrt{3}}{6}\mathbf{i})\right)^3$. *It maps* $t = 0, 1, d$ *to* $z = 1$ *and* $t = q/(\alpha\beta)$ *to* $z = 0$, *thus creating a new singular point.*

   If additionally $\gamma = \delta = \varepsilon = \frac{2}{3}$, *then* $P(t)$ *can be chosen also as degree-*6 *polynomial*

$$4\left(\left(1 - \frac{t}{\frac{1}{2} + \frac{\sqrt{3}}{6}\mathbf{i}}\right)^3 - \frac{1}{2}\right)^2,$$

   which maps $t = 0, 1, d, q/(\alpha\beta)$ *to* $z = 1$.

4. $d \in \mathcal{O}_{\mathfrak{S}_4}(\frac{1}{2} + \frac{5\sqrt{2}}{4}\mathbf{i})$. *Suppose* $d = \frac{1}{2} + \frac{5\sqrt{2}}{4}\mathbf{i}$, *then* $q/(\alpha\beta)$ *must be equal* $\frac{1}{2} + \frac{\sqrt{2}}{4}\mathbf{i}$, *characteristic exponents of the point* $t = d$ *must be* $0, 1/3$, *i.e.,* $\varepsilon = 2/3$, *and points* $t = 0, 1$ *must have characteristic exponents* $0, 1/2$, *i.e.,* $\gamma = \delta = 1/2$. *The function* $P(t)$ *is a degree-*4 *polynomial and can be chosen as*

$$\left(1 - \frac{t}{\frac{1}{2} + \frac{5\sqrt{2}}{4}\mathbf{i}}\right)\left(1 - \frac{t}{\frac{1}{2} + \frac{\sqrt{2}}{4}\mathbf{i}}\right)^3.$$

   It maps $t = 0, 1$ *to* $z = 1$ *and* $t = d, q/(\alpha\beta)$ *to* $z = 0$.

5. $d \in \mathcal{O}_{\mathfrak{S}_4}(\frac{1}{2} + \frac{11\sqrt{15}}{90}\mathbf{i})$. *Suppose* $d = \frac{1}{2} + \frac{11\sqrt{15}}{90}\mathbf{i}$, *then* $q/(\alpha\beta)$ *must be equal* $\frac{1}{2} + \frac{\sqrt{15}}{18}\mathbf{i}$, *characteristic exponents of the point* $t = d$ *must be* $0, 1/2$, *i.e.,* $\varepsilon = 1/2$, *and the points* $t = 0, 1$ *must have characteristic exponents* $0, 1/3$, *i.e.,* $\gamma = \delta = 2/3$. *The function* $P(t)$ *is a degree-*5 *polynomial and can be chosen as*

$$-\mathbf{i}\frac{2025\sqrt{15}}{64}t(t-1)\left(t - \frac{1}{2} - \frac{\sqrt{15}}{18}\mathbf{i}\right)^3.$$

   It maps $t = 0, 1, q/(\alpha\beta)$ *to* $z = 0$ *and* $t = d$ *to* $z = 1$.

Note that there are three independent parameters in the first case of Theorem B.1 (for example: α, β, γ) and all other cases contain only one or two free parameters. This means that the first case is more rife in applications. In fact, it is the only one, which occurs in the present book.

C

Orthogonal Complex Lie Algebras and Their Representations

C.1 Lie Algebra $\mathfrak{B}_k$

Here is a brief description of the simple complex Lie algebra $\mathfrak{B}_k \cong \mathfrak{so}(2k+1,\mathbb{C})$ (see [53, 60] and [135] for details).

Denote

$$S_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}(i),\ i \in \mathbb{N}\,.$$

Consider the Lie algebra $\mathfrak{B}_k \cong \mathfrak{so}(2k+1,\mathbb{C})$ as

$$\mathfrak{B}_k = \left(A \in \mathfrak{gl}(2k+1,\mathbb{C}) |\ A^T S_{2k+1} + S_{2k+1} A = 0 \right)\ . \tag{C.1}$$

Following [125], we shall enumerate the rows and columns of $A \in \mathfrak{B}_k$ by the indices $-k,\dots,-1,0,1,\dots,k$. The convenience of such notations is due to the fact that subalgebras $\mathfrak{B}_i \subset \mathfrak{B}_k$, $i<k$, correspond to indices of rows and columns from $-i$ to i.

It can be easily shown that a matrix

$$A = \sum_{i,j} a_{ij} E_{ij} \in \mathfrak{gl}(2k+1,\mathbb{C})$$

belongs to $\mathfrak{B}_k$ iff it holds $a_{ij} + a_{-j,-i} = 0$. This means that A is skew-symmetric w.r.t. its secondary diagonal.

Let $F_{ij} = E_{ij} - E_{-j,-i}$, then it is easily seen that

$$[F_{ij}, F_{pq}] = \delta_{jp} F_{iq} - \delta_{iq} F_{pj} + \delta_{-pi} F_{-qj} + \delta_{-jq} F_{p,-i}\ .$$

The algebra $\mathfrak{B}_k$ is spanned by elements F_{ij} with $i > -j$. Evidently, $F_{i,-i} = 0$ and $F_{-j,-i} = -F_{ij}$.

Elements F_{ii}, $i = 1,\dots,k$ form a base of the Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{B}_k$, which consists of elements of the form

$$X = \operatorname{diag}(-x_k, -x_{k-1}, \dots, -x_1, 0, x_1, \dots, x_{k-1}, x_k) \ .$$

Let $\varepsilon_i \in \mathfrak{h}_k^*$ such that $\varepsilon_i(X) = x_i$, i.e., ε_i is a base in $\mathfrak{h}_k^*$ dual to F_{ii}, $i = 1, \dots, k$. Define a symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{B}_k$ as

$$\langle A, B \rangle = \frac{1}{2} \operatorname{tr} AB \ , \tag{C.2}$$

which is proportional to the Killing form. Clearly,

$$\langle F_{ij}, F_{qp} \rangle = \delta_{ip}\delta_{jq}, \ i > -j, q > -p \ .$$

In particular, F_{ii}, $i = 1, \dots, k$, is an orthogonal base in $\mathfrak{h}_k$.

The form $\langle \cdot, \cdot \rangle|_{\mathfrak{h}_k}$ generates the isomorphism $\varkappa : \ \mathfrak{h}_k \to \mathfrak{h}_k^*$ by the formula $\varkappa(X) = \langle X, \cdot \rangle$. Specifically, $\varkappa(F_{i,i}) = \varepsilon_i$ and ε_i, $i = 1, \dots, k$ is an orthonormal base in $\mathfrak{h}_k^*$ w.r.t. the form

$$\langle f_1, f_2 \rangle^* := \langle \varkappa^{-1}(f_1), \varkappa^{-1}(f_2) \rangle, \ f_1, f_2 \in \mathfrak{h}_k^* \ .$$

Using this notation one can describe the standard form of the root system for $\mathfrak{B}_k$ in the following way. Let

$$\Phi_{\mathfrak{B}_k} := (\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j | \ i \neq j, \ i, j = 1, \dots, k)$$

be a root system in $\mathfrak{B}_k$,

$$\Phi_{\mathfrak{B}_k}^+ := (\varepsilon_i, \varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j | \ i > j, \ i, j = 1, \dots, k)$$

be a system of positive roots and

$$\Delta_{\mathfrak{B}_k} := (\alpha_1 = \varepsilon_1, \alpha_i = \varepsilon_i - \varepsilon_{i-1} | \ i = 2, \dots, k)$$

be a system of simple roots, corresponding to the inverse lexicographic order. A subalgebra $\mathfrak{B}_i \subset \mathfrak{B}_k$, $i < k$, corresponds to root systems $\Phi_{\mathfrak{B}_i}$, $\Phi_{\mathfrak{B}_i}^+$ and $\Delta_{\mathfrak{B}_i}$.

Let L_α be a root subspace in $\mathfrak{B}_k$, corresponding to a root $\alpha \in \Phi_{\mathfrak{B}_k}$. Then it holds

$$\begin{gathered} L_{-\varepsilon_i} = \operatorname{span}(F_{0i}), \ L_{\varepsilon_i} = \operatorname{span}(F_{i0}), \ L_{\varepsilon_i - \varepsilon_j} = \operatorname{span}(F_{ij}), \\ L_{\varepsilon_i + \varepsilon_j} = \operatorname{span}(F_{i,-j}), \ L_{-\varepsilon_i - \varepsilon_j} = \operatorname{span}(F_{-ij}), \ i, j = 1, \dots, k \ . \end{gathered} \tag{C.3}$$

Fundamental weights for $\mathfrak{B}_k$ are

$$\lambda_1 = \frac{1}{2} \sum_{j=1}^{k} \varepsilon_j, \ \lambda_i = \sum_{j=i}^{k} \varepsilon_j, \ i = 2, \dots, k \ .$$

Let

$$\lambda = \sum_{j=1}^{k} \lambda^j \lambda_j, \ \lambda^j \in \mathbb{Z}_+ := (0) \cup \mathbb{N}$$

be a dominant weight and $V(\lambda)$ be an irreducible finite-dimensional $\mathfrak{B}_k$-module with the highest weight λ. All finite-dimensional irreducible representations of $\mathfrak{B}_k$ are of this form, modules $V(\lambda)$ with different λ are not isomorphic to each other and $V(\lambda)$ corresponds to a (single-valued) representation of the group SO$(2k+1)$ iff λ_1 is even. The dominant weight λ can be written in the form

$$\lambda = \sum_{i=1}^{k} m_i \varepsilon_i,\ m_k \geqslant m_{k-1} \geqslant \ldots \geqslant m_1 \geqslant 0\ , \tag{C.4}$$

where either all $m_i \in \mathbb{Z}_+$ or all $m_i \in \mathbb{Z}_+ + \frac{1}{2}$. Even values of λ_1 corresponds to $m_i \in \mathbb{Z}_+$. Let δ be the sum of fundamental weights. Then it holds

$$\delta = \sum_{i=1}^{k} \lambda_i = \frac{1}{2} \sum_{\alpha \in \Phi^+_{\mathfrak{B}_k}}^{k} \alpha = \sum_{i=1}^{k} \left(i - \frac{1}{2}\right) \varepsilon_i\ . \tag{C.5}$$

The universal Casimir operator $C \in U(\mathfrak{B}_k)$ is

$$C = \sum_{i=1}^{k} \left(F_{ii}^2 + \{F_{i0}, F_{0i}\}\right) + \sum_{i>j>0} \left(\{F_{ij}, F_{ji}\} + \{F_{i,-j}, F_{-ji}\}\right)\ . \tag{C.6}$$

The following formulas are valid for any semisimple Lie algebra:

$$C|_{V(\lambda)} = (\langle \delta + \lambda, \delta + \lambda \rangle - \langle \delta, \delta \rangle)\, \mathrm{id}, \tag{C.7}$$

$$\dim V(\lambda) = \prod_{\alpha \succ 0} \langle \lambda + \delta, \alpha \rangle \Big/ \prod_{\alpha \succ 0} \langle \delta, \alpha \rangle, \tag{C.8}$$

where $\alpha \succ 0$ means a positive root.

For any semisimple Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$, the module $V(\lambda)$ can be decomposed into the finite direct sum of weight subspaces

$$V(\lambda) = \bigoplus_{\mu} V_\mu(\lambda),\ \mu \in \mathfrak{h}^*\ ,$$

where for $\forall v \in V_\mu(\lambda), \forall h \in \mathfrak{h}$, it holds $h(v) = \mu(h)v$ and the sum is over weights of the form

$$\lambda - \sum_{\alpha \succ 0} i_\alpha \alpha,\ i_\alpha \in \mathbb{Z}_+\ .$$

Besides, for any root α of $\mathfrak{g}$ one has

$$\xi_\alpha : V_\mu(\lambda) \to V_{\mu+\alpha}(\lambda)),\ \xi_\alpha \in L_\alpha\ . \tag{C.9}$$

C.2 Lie Algebra $\mathfrak{D}_k$

The Lie algebra $\mathfrak{D}_k$ is the subalgebra of $\mathfrak{B}_k$, consisting of matrices whose column and rows with the index 0 vanish. We shall discard these null row and column and shall enumerate other rows and columns of $A \in \mathfrak{D}_k$ by the indices $-k, \ldots, -1, 1, \ldots, k$ as before. The Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{D}_k$ is the same as in the $\mathfrak{B}_k$-case. Describe the $\mathfrak{D}_k$-case briefly, emphasizing differences from the $\mathfrak{B}_k$-case.

Now one has

$$\begin{aligned} \Phi_{\mathfrak{D}_k} &:= (\pm\varepsilon_i \pm \varepsilon_j |\ i \neq j,\, i,j = 1, \ldots, k)\,, \\ \Phi^+_{\mathfrak{D}_k} &:= (\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j |\ i > j,\, i,j = 1, \ldots, k)\,, \\ \Delta_{\mathfrak{D}_k} &:= (\alpha_1 = \varepsilon_1 + \varepsilon_2, \alpha_i = \varepsilon_i - \varepsilon_{i-1} |\ i = 2, \ldots, k)\ . \end{aligned}$$

The root subspaces $L_{\pm\varepsilon_i \pm \varepsilon_j}$ are the same as in $\mathfrak{B}_k$-case.

Fundamental weights are

$$\lambda_1 = \frac{1}{2}\sum_{j=1}^{k} \varepsilon_j,\ \lambda_2 = -\frac{1}{2}\varepsilon_1 + \frac{1}{2}\sum_{j=2}^{k} \varepsilon_j,\ \lambda_i = \sum_{j=i}^{k} \varepsilon_j,\ i = 3, \ldots, k\ .$$

The sum of fundamental weights is

$$\delta = \sum_{i=1}^{k} \lambda_i = \frac{1}{2} \sum_{\alpha \in \Phi^+_{\mathfrak{D}_k}} \alpha = \sum_{i=2}^{k} (i-1)\varepsilon_i\ .$$

A dominant weight

$$\lambda = \sum_{j=1}^{k} \lambda^j \lambda_j,\ \lambda^j \in \mathbb{Z}_+ := (0) \cup \mathbb{N}$$

now has the form

$$\lambda = \sum_{i=1}^{k} m_i \varepsilon_i,\ m_k \geqslant m_{k-1} \geqslant \ldots \geqslant m_2 \geqslant |m_1|, \tag{C.10}$$

where either $m_1 \in \mathbb{Z}, m_i \in \mathbb{Z}_+,\ i \geqslant 2$ or $m_1 \in \mathbb{Z} + \frac{1}{2}, m_i \in \mathbb{Z}_+ + \frac{1}{2},\ i \geqslant 2$. Again $\mathfrak{D}_k$-modules with integer $m_j,\ j = 1, \ldots, k$ correspond to (single-valued) representations of the group $\mathrm{SO}(2k)$.

The universal Casimir operator $C \in U(\mathfrak{D}_k)$ is

$$C = \sum_{i=1}^{k} F_{ii}^2 + \sum_{i>j>0} (\{F_{ij}, F_{ji}\} + \{F_{i,-j}, F_{-ji}\})\ . \tag{C.11}$$

C.3 Restrictions of $\mathfrak{B}_k$ and $\mathfrak{D}_k$-Representations

The following results were found in [211] (see also [212]).

Let $V_{\mathfrak{B}_k}(\lambda)$ be a simple $\mathfrak{B}_k$-module with a highest weight (C.4) and $V_{\mathfrak{D}_k}(\lambda)$ be a simple $\mathfrak{D}_k$-module with a highest weight

$$\lambda' = \sum_{i=1}^{k} m_i' \varepsilon_i,\ m_k' \geqslant m_{k-1}' \geqslant \ldots \geqslant m_2' \geqslant |m_1'|\ .$$

Proposition C.1. *The restriction $V_{\mathfrak{B}_k}(\lambda)|_{\mathfrak{D}_k}$ of the irreducible $\mathfrak{B}_k$-representation onto any subalgebra $\mathfrak{D}_k \subset \mathfrak{B}_k$ expands as follows*

$$V_{\mathfrak{B}_k}(\lambda)|_{\mathfrak{D}_k} = \bigoplus_{\lambda'} V_{\mathfrak{D}_k}(\lambda') ,$$

where the summation is over all λ' such that

$$m_k \geqslant m'_k \geqslant m_{k-1} \geqslant \ldots \geqslant m'_2 \geqslant m_1 \geqslant m'_1 \geqslant -m_1$$

and all m'_j are integer or half integer simultaneously with m_j.

Let $V_{\mathfrak{B}_{k-1}}(\lambda')$ be a simple $\mathfrak{B}_{k-1}$-module with a highest weight

$$\lambda' = \sum_{i=1}^{k-1} m'_i \varepsilon_i ,\ m'_{k-1} \geqslant m'_{k-2} \geqslant \ldots \geqslant m'_2 \geqslant m'_1 \geqslant 0 .$$

Proposition C.2. *The restriction $V_{\mathfrak{D}_k}(\lambda)|_{\mathfrak{B}_{k-1}}$ of the irreducible $\mathfrak{D}_k$-representation onto any subalgebra $\mathfrak{B}_{k-1} \subset \mathfrak{D}_k$ expands as follows*

$$V_{\mathfrak{D}_k}(\lambda)|_{\mathfrak{B}_{k-1}} = \bigoplus_{\lambda'} V_{\mathfrak{B}_{k-1}}(\lambda') ,$$

where the summation is over all λ' such that

$$m_k \geqslant m'_{k-1} \geqslant m_{k-1} \geqslant \ldots \geqslant m_2 \geqslant m'_1 \geqslant |m_1|$$

and all m'_j are integer or half integer simultaneously with m_j.

C.4 The Proof of Two Expansions

Expansions (8.14), (8.20) were obtained in [125] and [124] using the theory of Yangians, which is a part of the quantum algebra. Here we give an independent proof of these expansions from a classical point of view using one result from [41]. The initial idea of this proof is due to A.I. Molev.

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$, generated by elements Z_+, Z_-, F and relations

$$[F, Z_+] = 2Z_+,\ [F, Z_-] = -2Z_-,\ [Z_+, Z_-] = -\frac{1}{2}F^3 + qF,\ q \in \mathbb{C}. \quad \text{(C.12)}$$

Let also $\tau : \mathcal{A} \to \mathrm{Hom}_{\mathbb{C}}(V)$ be its irreducible linear representation in a finite-dimensional complex linear space V. An arbitrary linear operator in V has at least one eigenvalue. Let $F\chi = \eta\chi$, $\eta \in \mathbb{C}$, $\chi \in V, \chi \neq 0$, then relations (C.12) imply $FZ_+\chi = (\eta + 2)Z_+\chi$, $FZ_-\chi = (\eta - 2)Z_-\chi$. Since $\dim_C V < \infty$, there is a vector $v_\nu \in V$, $v_\nu \neq 0$ such that $Fv_\nu = \nu v_\nu$, $Z_+ v_\nu = 0$, $\nu \in \mathbb{C}$. Let $v_{\nu-2j} := Z_-^j v_\nu$, $j \in \mathbb{Z}_+$, then

$$Fv_\eta = \eta v_\eta,\ Z_- v_\eta = v_{\eta-2},\ \forall \eta \in L_\nu := \nu - 2\mathbb{Z}_+ . \quad \text{(C.13)}$$

Let μ be a root of the equation

$$\mu^2 + 2\mu + \nu^2 + 2\nu - 4q = 0 . \quad \text{(C.14)}$$

Lemma C.1. *It holds*

$$Z_+ v_\eta = \frac{1}{16}(\eta-\mu)(\eta-\nu)(\eta+\mu+2)(\eta+\nu+2)v_{\eta+2},\ \eta \in L_\nu\ . \tag{C.15}$$

Proof. For $\eta = \nu$ equality (C.15) is obvious. Let $\alpha \in L_\nu$, $\alpha \leqslant \nu$ and suppose that (C.15) is valid for any $\eta \in L_\nu$ such that $\eta > \alpha$. Then one gets

$$\begin{aligned} Z_+ v_\alpha &= Z_+ Z_- v_{\alpha+2} = [Z_+, Z_-]v_{\alpha+2} + Z_- Z_+ v_{\alpha+2} = \left(-\frac{1}{2}F^3 + qF\right) v_{\alpha+2} \\ &+ \frac{1}{16}(\alpha+2-\mu)(\alpha+2-\nu)(\alpha+\mu+4)(\alpha+\nu+4)Z_- v_{\alpha+4} \\ &= \left(-\frac{1}{2}(\alpha+2)^3 + q(\alpha+2)\right. \\ &\left. +\frac{1}{16}(\alpha+2-\mu)(\alpha+2-\nu)(\alpha+\mu+4)(\alpha+\nu+4)\right) v_{\alpha+2} \\ &= \frac{1}{16}(\alpha-\mu)(\alpha-\nu)(\alpha+\mu+2)(\alpha+\nu+2)v_{\alpha+2}, \end{aligned}$$

due to (C.12), (C.13), (C.15), and the following identity

$$\begin{aligned} &(\alpha+2-\mu)(\alpha+2-\nu)(\alpha+\mu+4)(\alpha+\nu+4) \\ -&(\alpha-\mu)(\alpha-\nu)(\alpha+\mu+2)(\alpha+\nu+2) = 8(\alpha+2)^3 - 16q(\alpha+2)\ . \end{aligned}$$

This completes the induction. □

Due to (C.13) nonzero vectors v_η, $\eta \in L_\nu$ are linearly independent, therefore the inequality $\dim_{\mathbb{C}} V < \infty$ yields $v_{\nu_1} \neq 0$, $v_{\nu_1-2} = 0$ for some $\nu_1 \in L_\nu$. Since the representation τ is irreducible, one gets

$$V = \operatorname{span}(v_{\nu_1}, v_{\nu_1+2}, \ldots, v_\nu)\,,\ \dim_{\mathbb{C}} V = \frac{1}{2}(\nu-\nu_1)+1 \tag{C.16}$$

due to (C.13) and (C.15). Equation (C.15) for $\eta = \nu_1 - 2$ implies

$$(\nu_1-2-\mu)(\nu_1+\mu)(\nu_1+\nu) = 0\ . \tag{C.17}$$

Let now $Z_+ = D^+, Z_- = D^-, F = F_{kk}$, $k \geqslant 2$ for the operators D^+, D^-, F_{kk} from Sect. 8.2.1, $V := \widetilde{V}_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})$ and therefore

$$q = \frac{1}{2}\left(m_k^2 + m_{k-1}^2 + (2k-1)m_k + (2k-3)m_{k-1}\right) + \frac{1}{4}(2k-1)(2k-3)\ . \tag{C.18}$$

Let $\mathcal{A}' := (U(\mathfrak{so}(2k+1,\mathbb{C})))^{\mathfrak{so}(2k-1,\mathbb{C})}$ be the centralizer in $U(\mathfrak{so}(2k+1,\mathbb{C}))$ of a Lie subalgebra $\mathfrak{so}(2k-1,\mathbb{C}) \subset \mathfrak{so}(2k+1,\mathbb{C})$. Obviously, the algebra $\mathcal{A}'$ acts in V. From Theorem 9.1.12 in [41] it follows that this action is irreducible. The definition of operators D^+, D^-, F_{kk}, $k \geqslant 2$ in Sect. 8.2.1 and Theorem 2.3 imply that algebras $\mathcal{A}'$ and $\mathcal{A}$ coincide modulo operators acting as scalars in the space V. Thus, the $\mathcal{A}$-module V is irreducible and Lemma C.1 is applicable.

We shall demonstrate that

$$(\nu_1 - 2 - \mu)(\nu_1 + \mu) \neq 0 \, . \tag{C.19}$$

Indeed the $\mathcal{A}$-module $\widetilde{V}_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})$ does not contain weight subspaces corresponding to weights $j\varepsilon_k$ for $|j| > m_k$ [60], therefore

$$(\nu_1 - 1)^2 + (\nu + 1)^2 \leqslant 2(m_k + 1)^2 \, . \tag{C.20}$$

If $(\nu_1 - 2 - \mu)(\nu_1 + \mu) = 0$ then $|\nu_1 - 1| = |\mu + 1|$ and (C.14) leads to the inequality

$$2 + 4q = (\mu + 1)^2 + (\nu + 1)^2 \leqslant 2(m_k + 1)^2 \, ,$$

which is equivalent to

$$m_{k-1}^2 + (2k-3)(m_k + m_{k-1}) + \frac{1}{2}(2k-1)(2k-3) \leqslant 0 \tag{C.21}$$

due to (C.18). Obviously inequality (C.21) is impossible.

Thus, from (C.16), (C.17), (C.19) and (8.8) one gets $\nu = -\nu_1 = m_k - m_{k-1}$. Now expansion (8.14) follows from (C.13) and (C.16).

The consideration for the space $V := \widetilde{V}_{\mathfrak{D}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})$, $k \geqslant 2$ and the operators $Z_+ = D^+, Z_- = D^-, F = F_{kk}$, $k \geqslant 2$ from Sect. 8.2.2 is similar. The $\mathcal{A}$-module V is irreducible due to the same reasons as above. Here one has

$$q = \frac{1}{2}\left(m_k^2 + m_{k-1}^2 + 2(k-1)m_k + 2(k-2)m_{k-1}\right) + (k-1)(k-2)$$

and the conjecture $(\nu_1 - 2 - \mu)(\nu_1 + \mu) = 0$ now implies by (C.20)

$$2(k-2)m_k + (m_{k-1} + k - 2)^2 + k(k-2) \leqslant 0$$

that leads to $k = 2$, $m_{k-1} = 0$ and to the equality in (C.20). This yields

$$\nu = -\nu_1 = m_k - |m_{k-1}| \, . \tag{C.22}$$

The last possibility $\nu_1 = -\nu$ in (C.17) also leads to (C.22) due to (8.18) and (C.16). Now expansion (8.20) is a consequence of (C.13), (C.16) and (C.22).

D

Unsolved Problems

1. Find nonconstant central potentials (or prove their absence), for which the quantum or classical two-body problem on a two-point homogeneous Riemannian space is integrable in some sense.
2. Prove the completeness of the generator system $D_0, \cdots, D_9$ for the algebra
$$\mathrm{Diff}_I\left(\mathbf{P}^2(\mathbb{C}a)_{\mathbf{S}}\right) .$$
3. Classify reduced phase spaces for the two-body problem on spaces
$$\mathbf{P}^n(\mathbb{C}), \mathbf{P}^n(\mathbb{H}), \mathbf{P}^2(\mathbb{C}a), \mathbf{H}^n(\mathbb{C}), \mathbf{H}^n(\mathbb{H}), \mathbf{H}^2(\mathbb{C}a),\ n = 2, 3 .$$
4. The discrete spectrum of the two-body problem in Euclidean space is obtained after the center of mass separation. It is interesting to find an alternative procedure, valid also for noncompact two-point homogeneous spaces (see Sect. 8.4). It is not contradict to conjecture 8.1, since the center of mass separation changes a Hamiltonian.

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